

## Bosonization of strongly interacting one-dimensional electrons

K. A. Matveev,<sup>1</sup> A. Furusaki,<sup>2</sup> and L. I. Glazman<sup>3</sup>

<sup>1</sup>*Materials Science Division, Argonne National Laboratory, Argonne, Illinois 60439, USA*

<sup>2</sup>*Condensed Matter Theory Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*

<sup>3</sup>*Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA*

(Received 1 August 2007; published 31 October 2007)

Strong repulsive interactions in a one-dimensional electron system suppress the exchange coupling  $J$  of electron spins to a value much smaller than the Fermi energy  $E_F$ . The conventional theoretical description of such systems based on the bosonization approach and the concept of Tomonaga-Luttinger liquid is applicable only at energies below  $J$ . In this paper, we develop a theoretical approach valid at all energies below the Fermi energy, including a broad range of energies between  $J$  and  $E_F$ . The method involves bosonization of the charge degrees of freedom, while the spin excitations are treated exactly. We use this technique to calculate the spectral functions of strongly interacting electron systems at energies in the range  $J \ll \varepsilon \ll E_F$ . We show that in addition to the expected features at the wave vector  $k$  near the Fermi point  $k_F$ , the spectral function has a strong peak centered at  $k=0$ . Our theory also provides analytical description of the spectral function singularities near  $3k_F$  (the “shadow band” features).

DOI: 10.1103/PhysRevB.76.155440

PACS number(s): 71.10.Pm

### I. INTRODUCTION

Recent experiments with quantum wires<sup>1–11</sup> and carbon nanotubes<sup>12,13</sup> have stimulated theoretical interest in transport properties of one-dimensional systems of interacting electrons. It is now widely accepted that in one dimension, interacting electrons form the so-called Luttinger liquid.<sup>14,15</sup> The main signature of the Luttinger liquid—the power-law behavior of the tunneling density of states—has recently been observed in experiments.<sup>11–13,16</sup> Another well-known prediction<sup>17–19</sup> of the Luttinger-liquid theory is that the conductance of a quantum wire connecting two-dimensional leads should be quantized in units of  $2e^2/h$ , regardless of the interaction strength. Although the quantization of conductance is routinely observed in modern experiments, careful recent measurements show significant deviations<sup>3–10</sup> from perfect quantization in the regime of very low electron density, where the effective electron-electron interactions are very strong.

The applicability of the Luttinger-liquid theory is not expected to be limited to weak interactions. On the other hand, the properties of the system do change significantly when the interactions become strong. It is well known that at low energies, one-dimensional electron systems support separate charge and spin excitation modes propagating at different velocities,<sup>20</sup>  $v_\rho$  and  $v_\sigma$ . Accordingly, the Luttinger-liquid theory describes the low-energy excitations of the system by two bosonic fields with linear dispersion, propagating at velocities  $v_\rho$  and  $v_\sigma$ . The applicability of such a description is limited to energies small compared to the bandwidths of the charge and spin excitations  $D_{\rho,\sigma} \sim \hbar v_{\rho,\sigma} n$ , where  $n$  is the electron density. In the noninteracting case, both velocities coincide with the Fermi velocity  $v_F$ , so  $D_\rho = D_\sigma \sim E_F$ . In the presence of weak repulsive interactions, the velocities are renormalized so that  $v_\sigma < v_\rho$ , but both velocities remain of order  $v_F$ . At strong interactions, the spin mode velocity  $v_\sigma$  is strongly suppressed,  $v_\sigma \ll v_\rho$ . In this case,  $D_\sigma \ll E_F \lesssim D_\rho$ , and the Luttinger-liquid theory is applicable only to phenomena

in which all the relevant energy scales are smaller than  $D_\sigma$ .

A number of recent theory papers<sup>21–30</sup> addressed the physics of strongly interacting electrons beyond the range of applicability of the Luttinger-liquid theory. Penc *et al.*<sup>21–23</sup> studied the tunneling density of states and spectral functions of the one-dimensional Hubbard model at energies in the range  $D_\sigma \ll \varepsilon \ll D_\rho$  and zero temperature. Cheianov and Zvonarev<sup>24,25</sup> and Fiete and Balents<sup>26</sup> explored the so-called spin-incoherent regime<sup>31</sup>  $D_\sigma \ll T \ll D_\rho$  and found an enhancement of the tunneling density of states at energies  $\varepsilon > T$ . Conductance of the quantum wire entering the spin-incoherent regime was predicted<sup>28,29</sup> to show behavior similar to the anomalies observed in experiments.<sup>3–10</sup>

Despite the recent theoretical successes in treating strongly interacting one-dimensional electrons, at present, there is no regular theoretical technique that can be applied to a broad class of problems and is not limited to the exactly solvable models.<sup>21–25</sup> In particular, the bosonization technique commonly used to justify the Luttinger-liquid picture<sup>14,15</sup> is applicable only at energies below the spin bandwidth  $D_\sigma$ . In this paper, we generalize the bosonization technique to all energies below the charge bandwidth  $D_\rho$ . Our method treats the spin excitations exactly, but applies bosonization to the charge excitations. It is thus applicable to a broad class of strong interactions, and is not limited to short-range coupling required for the existence of the exact solutions.

We apply our technique to the calculation of the spectral functions and the tunneling density of states of strongly interacting one-dimensional electron systems. Unlike many of the earlier treatments,<sup>21–25</sup> our calculations can be applied to systems with long-range interactions, such as quantum wires. In addition, our theory is valid in a broad range of temperatures: we obtain the spectral functions at zero temperature, in the spin-incoherent case  $T \gg D_\sigma$ , and also interpolate between these regimes. Furthermore, our approach provides a clear physical picture of the enhancement<sup>21,24–26</sup> of the tunneling density of states  $\nu(\varepsilon)$  at  $|\varepsilon| \ll E_F$ .

Our approach is introduced in Sec. II, where we derive the expression for the electron field operators at energies below  $D_\rho$  by bosonizing the charge modes while treating the spin excitations accurately, as excitations of an effective Heisenberg spin chain with a small exchange constant  $J$ . At energies below  $D_\sigma \sim J$ , the spin excitations can also be bosonized. This is accomplished in Sec. III, where we also demonstrate that the standard bosonization expression for the electron operators<sup>14,15</sup> is recovered in our approach when all important energy scales are below  $J$ . Calculation of various physical properties of the system requires knowledge of the electronic Green's functions discussed in Sec. IV. In the most interesting regime of energies,  $|\varepsilon| \gg J$ , the Green's functions are expressed in terms of certain equal-time correlation functions  $c^\pm(q)$  of the Heisenberg spin chain. Their behavior is important for understanding the electronic transport at  $|\varepsilon| \gg J$ ; it is discussed in Sec. IV. We calculate the spectral functions at energies  $J \ll |\varepsilon| \ll E_F$  in Sec. V and show that their dependence on the wave vector  $k$  has a Gaussian peak centered at  $k=0$ , which determines the behavior of the tunneling density of states. In addition, we find power-law singularities in the spectral functions at  $k$  near  $k_F$ , as well as the shadow band features<sup>22</sup> near  $3k_F$ . We conclude the paper with the discussion of our results in Sec. VI. A brief summary of some of our results was reported in Ref. 32.

## II. BOSONIZATION OF CHARGE EXCITATIONS

The most experimentally relevant one-dimensional system of strongly interacting electrons is realized in GaAs quantum wires.<sup>3-10</sup> In these devices, the spectrum of electrons is quadratic, and the system can be described by the standard Hamiltonian

$$H = -\frac{\hbar^2}{2m_e} \int \psi_\gamma^\dagger(x) \partial_x^2 \psi_\gamma(x) dx + \frac{1}{2} \int \int \psi_\gamma^\dagger(x) \psi_\beta^\dagger(y) V(x-y) \psi_\beta(y) \psi_\gamma(x) dx dy. \quad (1)$$

Here,  $m_e$  is the effective mass of electrons,  $\psi_\gamma(x)$  is the annihilation operator of electron with spin  $\gamma$  (summation over repeating spin indices is implied), potential  $V(x-y)$  describes the interaction between electrons, and  $\partial_x = d/dx$ .

The assumption of quadratic spectrum in Hamiltonian (1) is introduced for simplicity, and most of the results we obtain apply to a generic spectrum. In particular, our theory is applicable to the one-dimensional Hubbard model, with an important exception of the half-filled case, where the charge excitation spectrum is gapped.

The effect of interactions on the low-energy properties of the system is quantified by the parameters  $\eta_f = V(0)/\hbar v_F$  and  $\eta_b = V(2k_F)/\hbar v_F$ , where  $V(q)$  is the Fourier transform of the interaction potential. Parameter  $\eta_f$  controls the amplitude of forward scattering of two electrons at the Fermi surface. Positive value of  $\eta_f$  leads to the enhancement of the velocity of charge excitations  $v_\rho$  over the Fermi velocity. Parameter  $\eta_b$  controls the amplitude of backward scattering of two electrons. Strong backscattering impedes propagation of spin ex-

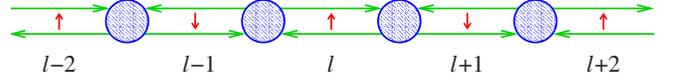


FIG. 1. (Color online) One-dimensional electrons at strong backscattering. Each electron is confined to the region of space between the neighboring particles. Shaded regions represent strong repulsive interactions.

citations through the system and leads to the suppression of the spin velocity  $v_\sigma$ . In the case of short-range interactions, the two parameters are of the same order of magnitude. On the other hand, in quantum wires, the electrons interact via long-range Coulomb repulsion, which is usually screened at a large distance  $d$  by a metal gate. In this case,  $\eta_f/\eta_b \sim \ln(nd)$ . Throughout this paper, we assume strong backscattering,  $\eta_b \gg 1$ .

### A. Effective Hamiltonian

In the limit  $\eta_b \rightarrow \infty$ , collisions of two electrons with opposite spins result in complete backscattering. In this case, the processes of spin exchange are completely suppressed, and the energy of the system no longer depends on the spin degrees of freedom. To find the energy of any state, one can assume that all the spins  $\gamma = \uparrow$  or, equivalently, assume that the fermions in Hamiltonian (1) are spinless,  $\psi_\gamma(x) \rightarrow \Psi(x)$ . The resulting Hamiltonian

$$H_\rho = -\frac{\hbar^2}{2m_e} \int \Psi^\dagger(x) \partial_x^2 \Psi(x) dx + \frac{1}{2} \int \int \Psi^\dagger(x) \Psi^\dagger(y) V(x-y) \Psi(y) \Psi(x) dx dy \quad (2)$$

describes the charge excitations in the system. Each eigenstate of Hamiltonian (2) for a system of  $N$  electrons is a degenerate multiplet of  $2^N$  spin states.

In this paper, the spinless fermions  $\Psi(x)$  will be referred to as *holons*. By construction, their number equals the total number of electrons,

$$\Psi^\dagger(x) \Psi(x) = \psi_\uparrow^\dagger(x) \psi_\uparrow(x) + \psi_\downarrow^\dagger(x) \psi_\downarrow(x). \quad (3)$$

In the limit of short-range coupling,  $V(x-y) = V_0 \delta(x-y)$ , the interactions in Hamiltonian (2) disappear due to the Pauli principle,  $\Psi(x) \Psi(x) = 0$ , and holons become free fermions. This fact is well known in the theory of the Hubbard model at strong interactions.<sup>33</sup>

The physical picture of one-dimensional electrons in the limit of strong backscattering is illustrated in Fig. 1. Due to the strong repulsive interactions, electrons cannot pass through each other. As a result, electron  $l$  is always confined between electrons  $l-1$  and  $l+1$ , and their spins cannot move through the system.

At strong but finite repulsion, the amplitude of forward scattering of electrons does not vanish. Such processes give rise to a weak exchange of the spins at the neighboring sites, e.g.,  $l$  and  $l+1$ . To leading order, the coupling of the next-nearest neighbors can be neglected. The symmetry with respect to spin rotations dictates the form of coupling between the spins:

$$H_\sigma = \sum_l JS_l \cdot S_{l+1}. \quad (4)$$

The exchange constant  $J$  is positive, as otherwise the ground state of a system of one-dimensional electrons would have been spin polarized, in violation of the Lieb-Mattis theorem.<sup>34</sup>

Thus, we conclude that the Hamiltonian of a one-dimensional system of strongly interacting electrons can be written as a sum  $H_\rho + H_\sigma$ . In the case of the Hubbard model, this was first noticed by Ogata and Shiba,<sup>33</sup> who discovered that in the limit of strong repulsion, the Bethe ansatz ground state of the system factorizes into a direct product of the ground state of noninteracting fermions (holons) and the ground state of the Heisenberg spin chain [Eq. (4)]. In the context of quantum wires, the description based on the Hamiltonian  $H_\rho + H_\sigma$  was used in Refs. 28 and 29. The long-range nature of the Coulomb repulsion between electrons in a quantum wire results in the exponential suppression of the exchange constant,<sup>29,35-37</sup>

$$J = J^* \exp\left(-\frac{2.80}{\sqrt{na_B}}\right). \quad (5)$$

Here, the prefactor  $J^* \sim E_F(na_B)^{-3/4}$ , the Bohr radius is defined as  $a_B = \epsilon\hbar^2/m_e e^2$ , and  $\epsilon$  is the dielectric constant.

It is worth mentioning that one can replace Hamiltonian (1) with the sum of two independent Hamiltonians (2) and (4) only at sufficiently low energies. Indeed, the backscattering amplitude for two electrons with wave vectors  $\pm k$  is  $V(2k)/\hbar v_k$  and tends to zero at  $k \rightarrow \infty$  for any reasonable interaction potential. (Here,  $v_k = \hbar k/m$  is the velocity of an electron with wave vector  $k$ .) Thus, at high energies, the initial assumption of strong backscattering is violated. To find the region of applicability of our low-energy theory, one can estimate the correction to exchange constant (5) caused by the fact that at higher energies, the distances between electrons fluctuate, and the density  $n$  is no longer constant. Given that the rigidity of the Wigner crystal is due to the Coulomb repulsion between the electrons, we estimate  $\delta n \sim \sqrt{m_e n a_B} |\epsilon|/\hbar$ , and the exchange  $J$  acquires significant corrections in the presence of excitations with energies  $|\epsilon| \gtrsim (\hbar n)^2/m_e \sim E_F$ . Thus, our subsequent results are valid up to energies of order  $E_F$ , rather than the somewhat higher energy scale  $D_\rho \sim E_F \sqrt{\ln(nd)/na_B}$ . In the case of short-range interactions, the scales  $D_\rho$  and  $E_F$  are of the same order of magnitude.

### B. Electron creation and destruction operators

Effective Hamiltonian (2), (4) is defined in terms of the holon field operators  $\Psi(x)$  and the spin operators  $S_l$ , rather than the original electron operators  $\psi_{\uparrow,\downarrow}(x)$ . In order to apply the effective theory (2), (4) to problems formulated in terms of electrons (e.g., calculation of the spectral functions), we need to establish the relations between the electron operators and the new variables.

In the context of the Hubbard model, this issue was addressed by Penc *et al.*,<sup>21</sup> who used the definition

$$\psi_\gamma^\dagger(0) = \Psi^\dagger(0)Z_{0,\gamma}^\dagger. \quad (6)$$

Given relation (3) between the densities of electrons and holons, creation of an electron at point 0 must be accompanied by creation of a holon. In addition, when a new particle is added to the system of  $N$  electrons, spin chain (4) acquires an additional site. This is accounted for by the operator  $Z_{0,\gamma}^\dagger$ . By definition, operator  $Z_{l,\gamma}^\dagger$  adds a new site with spin  $\gamma$  to the spin chain between the sites  $l-1$  and  $l$ .

Despite the fact that rule (6) leads to a number of correct results when applied carefully,<sup>21-23</sup> it is not a completely satisfactory expression of an electron creation operator in terms of the charge and spin degrees of freedom. In particular, the generalization of Eq. (6) to  $x \neq 0$  is not straightforward.<sup>22</sup> The origin of the difficulty lies in the fact that unlike the electrons and holons, the spins in Hamiltonian (4) are not assigned to specific points in space. When an electron is created at point  $x$ , the additional site in spin chain (4) appears at  $l=l(x)$ , where

$$l(x) = \int_{-\infty}^{x+0} \Psi^\dagger(y)\Psi(y)dy \quad (7)$$

is the number of electrons (or holons) between  $-\infty$  and point  $x$ . We shall therefore define the electron creation and annihilation operators as

$$\psi_\gamma^\dagger(x) = Z_{l(x),\gamma}^\dagger \Psi^\dagger(x), \quad (8a)$$

$$\psi_\gamma(x) = \Psi(x)Z_{l(x),\gamma}. \quad (8b)$$

The most important difference between expressions (6) and (8a) is that the latter explicitly accounts for the fact that the spins are attached to electrons. Thus, despite the apparent separation of the charge and spin degrees of freedom in effective Hamiltonian (2), (4) electron creation operator (8a) does not factorize into a product of two operators acting on only charge or only spin variables.<sup>38</sup> In Appendix A, we show that our operators (8) satisfy the appropriate anticommutation relations.

### C. Bosonization of holon operators

In this paper, we are interested in the properties of strongly interacting one-dimensional systems at energies well below the Fermi energy,  $|\epsilon| \ll E_F$ . In this case, the dynamics of the charge degrees of freedom described by Hamiltonian (2) simplifies dramatically. Indeed, it is well known that the low-energy properties of a system of interacting spinless fermions are accurately described by the Tomonaga-Luttinger model,<sup>14,15</sup>

$$H_\rho = \frac{\hbar v_\rho}{2\pi} \int [K(\partial_x \theta)^2 + K^{-1}(\partial_x \phi)^2] dx. \quad (9)$$

Here,  $\phi$  and  $\theta$  are bosonic fields satisfying the commutation relations  $[\phi(x), \partial_y \theta(y)] = \pi i \delta(x-y)$ . Field  $\theta$  is related to the momentum density of the system,  $p(x) = \hbar n \partial_x \theta(x)$ , whereas  $\phi$  is defined in terms of the density of fermions,

$$\Psi^\dagger(x)\Psi(x) = \frac{1}{\pi}[k_F^h + \partial_x\phi(x)]. \quad (10)$$

Here, we have introduced the Fermi momentum of the holons  $k_F^h$ . Considering that the total electron density equals the holon density, Eq. (3), we have  $n=2k_F/\pi=k_F^h/\pi$ . Thus, the Fermi wave vectors of holons and electrons are related by

$$k_F^h = 2k_F. \quad (11)$$

Finally, the parameter  $K \leq 1$  in Hamiltonian (9) is the so-called Luttinger-liquid constant.

Transformation of the Hamiltonian  $H_\rho$  from fermionic form (2) to the bosonic form (9) is accomplished via bosonization procedure,<sup>14,15</sup> in which the fermion operators  $\Psi$  are expressed in terms of the bosonic fields  $\phi$  and  $\theta$ . At the first step, one notices that at low energies, the properties of one-dimensional Fermi systems are dominated by excitations near the two Fermi points. Particles near each of the Fermi points propagate in one of two possible directions, right or left. Thus, the fermion operator is presented as a sum of two chiral fermions,

$$\Psi(x) = \Psi_R(x) + \Psi_L(x), \quad (12)$$

where the operators  $\Psi_R$  and  $\Psi_L$  destroy fermions with wave vectors near the right and left Fermi points, respectively.

The chiral fermion operators are bosonized following the prescription

$$\Psi_{R,L}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-i\theta(x)} e^{\pm i[k_F^h x + \phi(x)]}, \quad (13)$$

where  $\alpha$  is a short-distance cutoff.

Using relations (7), (10), (12), and (13), we express electron annihilation operator (8b) in the form

$$\psi_\gamma(x) = \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} (e^{i[k_F^h x + \phi(x)]} + e^{-i[k_F^h x + \phi(x)]}) Z_{l,\gamma}|_{l=(1/\pi)[k_F^h x + \phi(x)]}. \quad (14)$$

Unlike original formula (8b), this expression is valid only at low energies,  $|\varepsilon| \ll E_F$ . Its advantage is that the charge modes are now presented in the form of noninteracting bosons [Eq. (9)] and can be treated rather easily. We apply expression (14) to the calculation of the electron Green's functions at  $J, T \ll |\varepsilon| \ll E_F$  in Sec. IV.

### III. BOSONIZATION OF SPIN EXCITATIONS

As we discussed in Sec. I, the conventional Tomonaga-Luttinger theory<sup>14,15</sup> of low-energy properties of one-dimensional electron systems is based on the idea of bosonization of electron operators. Mathematically, this corresponds to applying procedure (13) to annihilation operators of electrons with spins  $\gamma = \uparrow, \downarrow$ ,

$$\psi_{R,\gamma}(x) = \frac{e^{ik_F x}}{\sqrt{2\pi\alpha}} e^{(i/\sqrt{2})[\phi_\rho(x) - \theta_\rho(x)]} e^{\pm(i/\sqrt{2})[\phi_\sigma(x) - \theta_\sigma(x)]}, \quad (15a)$$

$$\psi_{L,\gamma}(x) = \frac{e^{-ik_F x}}{\sqrt{2\pi\alpha}} e^{-(i/\sqrt{2})[\phi_\rho(x) + \theta_\rho(x)]} e^{\mp(i/\sqrt{2})[\phi_\sigma(x) + \theta_\sigma(x)]}. \quad (15b)$$

Here,  $\phi_{\rho,\sigma}$  and  $\theta_{\rho,\sigma}$  are the bosonic fields describing the charge and spin excitation modes of the system. In terms of these fields, the Hamiltonian of interacting electrons (1) takes its bosonized low-energy form  $H_\rho + H_\sigma$  with

$$H_\rho = \frac{\hbar v_\rho}{2\pi} \int [K_\rho (\partial_x \theta_\rho)^2 + K_\rho^{-1} (\partial_x \phi_\rho)^2] dx, \quad (16a)$$

$$H_\sigma = \frac{\hbar v_\sigma}{2\pi} \int [K_\sigma (\partial_x \theta_\sigma)^2 + K_\sigma^{-1} (\partial_x \phi_\sigma)^2] dx + \frac{2g_{1\perp}}{(2\pi\alpha)^2} \int \cos[\sqrt{8}\phi_\sigma(x)] dx. \quad (16b)$$

Here, the matrix element  $g_{1\perp}$  accounts for the processes of backscattering of two electrons with opposite spins. The respective sine-Gordon term in Hamiltonian (16b) is marginally irrelevant. In the absence of magnetic field, the SU(2) symmetry of the problem requires<sup>15</sup> that when  $g_{1\perp}$  scales to zero, the Luttinger parameter  $K_\sigma \rightarrow 1$ .

Hamiltonian (16) is typically derived under the assumption that the electron-electron interactions are weak. On the other hand, it represents a stable low-energy fixed point of the theory, and thus should be valid beyond the weak-interaction approximation. Although the nature of the low-energy fixed point can, in principle, change at a finite value of the interaction strength, such a change would imply a quantum phase transition, which is generally not expected. The more likely scenario is that Hamiltonian (16) is the correct low-energy description of one-dimensional electron systems at arbitrarily strong interactions. Under this assumption, one should expect to be able to show that at low energies (i) Hamiltonian (9), (4) is equivalent to Eq. (16) and (ii) at the same time our expression (14) for the electron destruction operator transforms to Eq. (15). We now show that this is indeed the case.

#### A. Low-energy Hamiltonian of strongly interacting electrons in one dimension

First, we notice that expressions (9) and (16a) for  $H_\rho$  are very similar, as they both describe acoustic excitations in the charge channel propagating at speed  $v_\rho$ . Although it is natural to assume that  $K$ ,  $\phi$ , and  $\theta$  in Eq. (9) should be identified with  $K_\rho$ ,  $\phi_\rho$ , and  $\theta_\rho$  in Eq. (16a), respectively, this is not the case. The correct approach is to ensure that the physically observable quantities, such as electron density (3), have equivalent expressions in both theories. In the standard bosonization description based on Eq. (15), the electron density is given by

$$\psi_\uparrow^\dagger(x)\psi_\uparrow(x) + \psi_\downarrow^\dagger(x)\psi_\downarrow(x) = \frac{2k_F}{\pi} + \frac{\sqrt{2}}{\pi} \partial_x \phi_\rho(x). \quad (17)$$

Comparing this expression with Eqs. (3), (10), and (11), we conclude

$$\phi(x) = \sqrt{2} \phi_\rho(x). \quad (18a)$$

Then, to preserve the proper commutation relations between bosonic fields, one has to assume

$$\theta(x) = \frac{1}{\sqrt{2}} \theta_\rho(x). \quad (18b)$$

Finally, substituting Eqs. (18a) and (18b) into Hamiltonian (9), we recover Eq. (16a) if

$$K = 2K_\rho. \quad (18c)$$

Turning to the spin Hamiltonian  $H_\sigma$ , we note that there is a well-known procedure<sup>15</sup> of bosonization of Heisenberg model (4). One starts by converting spin operators  $S_l$  to spinless fermion operators  $a_l$  via the Jordan-Wigner transformation

$$S_l^z = a_l^\dagger a_l - \frac{1}{2}, \quad S_l^x + iS_l^y = a_l^\dagger \exp\left(i\pi \sum_{j=1}^{l-1} a_j^\dagger a_j\right). \quad (19)$$

In terms of the spinless fermions, the Hamiltonian  $H_\sigma$  takes the form

$$H_\sigma = H^{xy} + H^z, \quad (20a)$$

$$H^{xy} = \frac{1}{2} \sum_l J (a_l^\dagger a_{l+1} + a_{l+1}^\dagger a_l), \quad (20b)$$

$$H^z = \sum_l J \left( a_l^\dagger a_l - \frac{1}{2} \right) \left( a_{l+1}^\dagger a_{l+1} - \frac{1}{2} \right). \quad (20c)$$

Thus, Heisenberg model (4) is reduced to the tight-binding model of spinless fermions with repulsive interactions between particles at the nearest-neighbor sites.

The steps leading from Eq. (4) to Eq. (20) are exact, and the spectra of the two Hamiltonians are identical at all energies. At energies much smaller than  $J$ , one can simplify Hamiltonian (20) by bosonizing the Jordan-Wigner fermions. One starts by considering the noninteracting model given by Eq. (20b). The spectrum of that Hamiltonian is obtained as the sum of energies  $\epsilon(q)$  of independent spinless fermions,

$$\epsilon(q) = J \cos ql, \quad (21)$$

where the wave vector  $q$  varies from 0 to  $2\pi$ .

In the absence of external magnetic field, one expects  $\langle S_l^z \rangle = 0$ . According to Eq. (19) band (21) is half-filled, and the two Fermi points are at

$$q_L = \frac{\pi}{2}, \quad q_R = \frac{3\pi}{2}. \quad (22)$$

The bosonization is accomplished by presenting the operator  $a_l$  as a sum of operators destroying the right- and left-moving particles,

$$a_l = a_R(l) + a_L(l), \quad (23)$$

where

$$a_R(\xi) = \frac{1}{\sqrt{2\pi\tilde{\alpha}}} e^{iq_R\xi} e^{i\varphi_R(\xi)}, \quad (24a)$$

$$a_L(\xi) = \frac{1}{\sqrt{2\pi\tilde{\alpha}}} e^{iq_L\xi} e^{-i\varphi_L(\xi)}. \quad (24b)$$

Since the bosonization description concentrates on the range of momenta close to the Fermi points, the discrete site number  $l$  is replaced here with the continuous coordinate  $\xi$ . (Unlike the coordinate  $x$  of electrons,  $\xi$  is dimensionless.) The chiral bosonic fields  $\varphi_R$  and  $\varphi_L$  satisfy the commutation relations

$$[\varphi_R(\xi), \partial_{\xi'} \varphi_R(\xi')] = -2\pi i \delta(\xi - \xi'), \quad (25a)$$

$$[\varphi_L(\xi), \partial_{\xi'} \varphi_L(\xi')] = 2\pi i \delta(\xi - \xi'), \quad (25b)$$

$$[\varphi_L(\xi), \partial_{\xi'} \varphi_R(\xi')] = 0. \quad (25c)$$

Upon bosonization (24), Hamiltonian (20b) takes the form

$$H^{xy} = \frac{J}{4\pi} \int [(\partial_\xi \varphi_R)^2 + (\partial_\xi \varphi_L)^2] d\xi. \quad (26)$$

The next step of the bosonization procedure is to convert to nonchiral bosonic fields

$$\varphi = \frac{1}{2}(\varphi_L + \varphi_R), \quad \vartheta = \frac{1}{2}(\varphi_L - \varphi_R). \quad (27)$$

As a result, Hamiltonian (26) takes the form

$$H_0 = \frac{\hbar\tilde{v}}{2\pi} \int \left[ \mathcal{K}(\partial_\xi \vartheta)^2 + \frac{1}{\mathcal{K}}(\partial_\xi \varphi)^2 \right] d\xi, \quad (28a)$$

with  $\tilde{v} = J/\hbar$  and  $\mathcal{K} = 1$ .

When interaction term (20c) is added to Hamiltonian (20b), and bosonization transformation (24) is applied, the parameters  $\tilde{v}$  and  $\mathcal{K}$  change, and an additional term appears in the Hamiltonian,

$$V = \frac{2g}{(2\pi\tilde{\alpha})^2} \int \cos[4\varphi(\xi)] d\xi, \quad (28b)$$

with  $g \sim 1$ .

Hamiltonian (28a) is equivalent to spin part (16b) of the Hamiltonian of weakly interacting electron system if one assumes

$$\varphi(\xi) = \frac{1}{\sqrt{2}} \phi_\sigma(\xi/n), \quad (29a)$$

$$\vartheta(\xi) = \sqrt{2} \theta_\sigma(\xi/n), \quad (29b)$$

$$\tilde{v} = v_\sigma n, \quad (29c)$$

$$\mathcal{K} = \frac{K_\sigma}{2}. \quad (29d)$$

In the absence of magnetic field, as cosine term (28b) scales to zero at low energies, the Luttinger parameter  $\mathcal{K}$  approaches  $1/2$ , as required by the SU(2) symmetry of the problem.<sup>15</sup> Thus, Eq. (29d) is consistent with the similar requirement  $K_\sigma \rightarrow 1$  in Hamiltonian (16b).

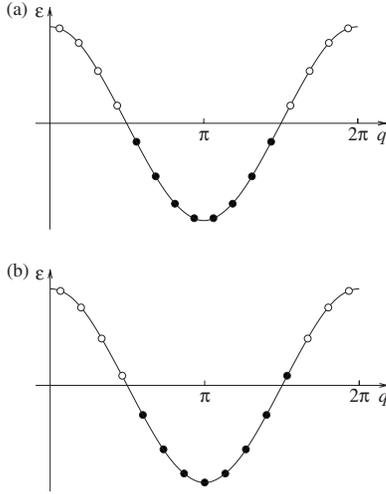


FIG. 2. Ground state of the system of noninteracting fermions [Eq. (20b)] for the case of 16 sites and 8 fermions. Filled circles show wave vectors and energies of the 8 occupied single-particle states. (b) Ground state of the same problem with 15 sites.

### B. Bosonization of the operators $Z_{l,\gamma}$

In order to demonstrate that in the regime of low energies  $|\epsilon| \ll J$  electron destruction operator (14) takes standard form (15), one needs to bosonize the operator  $Z_{l,\gamma}$  in Eq. (14). We start with  $Z_{0,\downarrow}$ . This operator acts on an arbitrary state in the Hilbert space of Hamiltonian (4) with  $N$  sites. If the spin at the site  $l=0$  is  $\downarrow$ , the operator  $Z_{0,\downarrow}$  removes that site from the spin chain; if the spin is  $\uparrow$ , the outcome is zero. The Jordan-Wigner transformation [Eq. (19)] defines one spinless fermion per each site with spin  $\uparrow$ . Thus, the operator  $Z_{0,\downarrow}$  removes site  $l=0$  from tight-binding model (20b) without changing the number of fermions  $N_f$ .

To derive the bosonized form of  $Z_{0,\downarrow}$ , it is convenient to consider its effect on the eigenstates of noninteracting model (20b). The latter are Slater determinants of plane waves with wave vectors in the range  $0 \leq q_j < 2\pi$  and energies  $\epsilon(q_j)$  given by Eq. (21). To determine the allowed values of wave vectors  $q_j$ , we assume periodic boundary conditions on the spin chain,  $S_0 = S_N$ . Because of the Jordan-Wigner string in definition (19) of the spinless fermions, their respective boundary conditions are either periodic or antiperiodic, depending on the parity of their number  $N_f$ ,

$$a_N^\dagger = (-1)^{N_f-1} a_0^\dagger. \quad (30)$$

Thus, the wave vectors of the fermions take the values

$$q_j = \begin{cases} \frac{2\pi}{N}j & \text{for odd } N_f \\ \frac{2\pi}{N}(j + \frac{1}{2}) & \text{for even } N_f, \end{cases} \quad (31)$$

where  $j=0, 1, \dots, N-1$ .

The ground state of Hamiltonian (20b) is illustrated in Fig. 2(a). Upon removal of one site from the chain, the allowed values of the wave vector increase slightly,

$$q_j \rightarrow q_j \frac{N}{N-1} \approx q_j + \frac{q_j}{N}. \quad (32)$$

The effect of the operator  $Z_{0,\downarrow}$  on the eigenstates of  $H^{\text{xy}}$  can be interpreted as follows. By removing a site from the spin chain,  $Z_{0,\downarrow}$  creates a scattering potential for the Jordan-Wigner fermions near  $\xi=0$ . Only forward scattering is present, and the wave functions of the fermions acquire phase shifts proportional to the wave vectors,

$$e^{iq\xi} \rightarrow e^{iq\xi} e^{-i(q/2)\text{sgn}(\xi)}. \quad (33)$$

When the periodic boundary conditions are imposed on the fermions, the phase shift in Eq. (33) moves the allowed values of the wave vectors by  $q/N$ , in agreement with Eq. (32).

In the bosonization treatment of the spin chain, one concentrates on the vicinities of the two Fermi points  $q_R$  and  $q_L$ , where the fermions are classified as either right or left moving, Eq. (23). According to Eq. (33), the operator  $Z_{0,\downarrow}$  transforms the fermion operators as

$$a_{R,L}(\xi) \rightarrow Z_{0,\downarrow} a_{R,L}(\xi) Z_{0,\downarrow}^\dagger = a_{R,L}(\xi) e^{-i(q_{R,L}/2)\text{sgn}(\xi)}. \quad (34)$$

Using bosonized representation (24) of the fermion operators and commutation relations (25), we conclude that

$$Z_{0,\downarrow} = \exp \left\{ -\frac{i}{2\pi} [q_L \varphi_L(0) + q_R \varphi_R(0)] \right\}. \quad (35)$$

Here, we omit a numerical prefactor, which depends on the specific cutoff procedure used in the bosonization scheme, but can be considered to be of order unity.

To find the bosonized expression for  $Z_{l,\downarrow}$  away from the point  $l=0$ , it is not sufficient to replace the arguments of  $\varphi_L$  and  $\varphi_R$  with  $l$ . Indeed, our derivation of Eq. (35) allowed for an arbitrary phase factor, which may depend on  $l$ . To determine this phase factor, we notice that

$$Z_{l,\gamma} = e^{-i\hat{Q}l} Z_{0,\gamma} e^{i\hat{Q}l}, \quad (36)$$

where  $\hat{Q}$  is the operator of the total momentum of the system. (This relation becomes clear if one notices that the operator  $e^{i\hat{Q}l}$  shifts the spin chain by  $l$  sites to the left.) Summing the changes of wave vectors (32) for all particles between Fermi points (22), we conclude that  $Z_{0,\sigma}$  increases  $Q$  by  $\pi/2$ . Thus, we obtain

$$Z_{l,\downarrow} = \exp \left[ -i\frac{\pi}{2}l - \frac{i}{4}\varphi_L(l) - \frac{3i}{4}\varphi_R(l) \right], \quad (37)$$

where we have also substituted into Eq. (35) the values of Fermi wave vectors (22).

The apparent asymmetry between the left and right movers in Eq. (37) can be understood by noticing that interpretation (32) of the changes of wave vectors of the fermions is not unique. Instead of assuming that as we remove a site from the spin chain, the fermion states in Fig. 2(a) transform into those of Fig. 2(b) by shifting to the right, Eq. (32), one can assume that all the states move to the left,

$$q_j \rightarrow q_j \frac{N}{N-1} - \frac{2\pi}{N-1} \simeq q_j + \frac{q_j - 2\pi}{N}. \quad (38)$$

As a result, the system arrives at a new ground state on the  $(N-1)$ -site lattice, which is the mirror image of the state shown in Fig. 2(b). One can repeat the above arguments leading to Eq. (35) and obtain the new expression by replacing  $q_{R,L} \rightarrow q_{R,L} - 2\pi$ . Similar to the presentation of the fermion operators  $a_l$  as a sum of two chiral contributions (23), we conclude that

$$Z_{l,\downarrow} = \exp \left[ -i \frac{\pi}{2} l - \frac{i}{4} \varphi_L(l) - \frac{3i}{4} \varphi_R(l) \right] + \exp \left[ i \frac{\pi}{2} l + \frac{3i}{4} \varphi_L(l) + \frac{i}{4} \varphi_R(l) \right]. \quad (39)$$

Apart from these two contributions, the operator  $Z_{l,\downarrow}$  may contain terms corresponding to greater shifts of the fermion states in the momentum space. They can be obtained by adding any multiples of  $2\pi/N$  to the right-hand side of Eq. (32). Such terms can be viewed as operator (35) combined with  $(a_L^\dagger a_R)^m$  or  $(a_R^\dagger a_L)^m$ . At low energies, such terms are less relevant than leading contributions (39) and can be neglected.

A similar bosonization procedure can be performed with operator  $Z_{l,\uparrow}$ . In addition to shifts (32) of the wave vectors of the allowed fermion states caused by the change of system size  $N \rightarrow N-1$ , one also needs to account for the change in the fermion number,  $N_f \rightarrow N_f - 1$ . The latter changes wave vector quantization conditions (31) and removes a particle from either the right or the left Fermi point. The two contributions resulting from such treatment are

$$Z_{l,\uparrow} = \exp \left[ -i \frac{\pi}{2} l + \frac{i}{4} \varphi_L(l) + \frac{3i}{4} \varphi_R(l) \right] + \exp \left[ i \frac{\pi}{2} l - \frac{3i}{4} \varphi_L(l) - \frac{i}{4} \varphi_R(l) \right]. \quad (40)$$

Replacing the chiral bosonic fields in Eqs. (39) and (40) with their nonchiral versions [Eq. (27)], we find

$$Z_{l,\gamma} = e^{i(\pi/2)l} e^{\mp i[\varphi(l)+(1/2)\vartheta(l)]} + e^{-i(\pi/2)l} e^{\pm i[\varphi(l)-(1/2)\vartheta(l)]}, \quad (41)$$

where the upper and lower signs correspond to  $\gamma = \uparrow$  and  $\downarrow$ , respectively. A generalization of bosonization rule (41) to the case of nonvanishing magnetization is discussed in Appendix B.

### C. Two-step bosonization procedure for the electron operators

In this paper, we consider one-dimensional electron systems with strong repulsive interactions, when the spin exchange between electrons is strongly suppressed,  $J \ll E_F$ . In such systems, the bosonization of electron operators  $\psi_\gamma(x)$  can be performed in two steps. At energies  $|\varepsilon|$  below the Fermi energy  $E_F$ , the charge excitations can be bosonized, and the fermion operators take form (14). This expression does not assume a specific relation between  $\varepsilon$  and  $J$ , so the

spin excitations are accounted for accurately by the operators  $Z_{l,\gamma}$ . On the other hand, if  $|\varepsilon| \ll J$ , the spin excitations can also be bosonized [Eq. (41)]. To compare the resulting expression for the electron destruction operators with those used in the standard bosonization procedure [Eq. (15)], we substitute Eq. (41) into Eq. (14). This substitution results in four terms in the expression for  $\psi_\gamma(x)$ . To identify the annihilation operator for the right-moving electron [Eq. (15a)], we combine the first term in the brackets in Eq. (14) with the first term in the right-hand side of Eq. (41). This yields

$$\begin{aligned} \psi_{R,\gamma}(x) &= \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} e^{i[k_F^h x + \phi(x)]} e^{-i(\pi/2)l} e^{\pm i[\varphi(l)-(1/2)\vartheta(l)]} \Big|_{l=(1/\pi)[k_F^h x + \phi(x)]} \\ &\simeq \frac{e^{(i/2)k_F^h x}}{\sqrt{2\pi\alpha}} e^{-i\theta(x)} e^{(i/2)\phi(x)} e^{\pm i[\varphi(nx)-(1/2)\vartheta(nx)]}. \end{aligned} \quad (42)$$

Expressing the bosonic fields via  $\phi_{\rho,\sigma}$  and  $\theta_{\rho,\sigma}$  with the help of Eqs. (18a), (18b), and (29), we find that our result (42) is equivalent to the standard expression [Eq. (15a)]. Similarly, combining the second term in the brackets in Eq. (14) with the second term in the right-hand side of Eq. (41), one reproduces the bosonized expression for the annihilation operator (15b) of the left-moving electrons.

To understand the meaning of the remaining two contributions to  $\psi_\gamma(x)$ , let us consider the momenta of the charge and spin excitations in Eq. (42). By destroying the right-moving holon, we change the momentum of the system by  $k_F^h = 2k_F$ . Thus, to obtain expression for annihilation operator of electron with momentum near the right Fermi point  $k_F$ , we chose the left-moving component of the operator  $Z_{l,\gamma}$ , which reduces the momentum change by  $k_F$ . By choosing the other component of  $Z_{l,\gamma}$ , we increase the total momentum change to  $3k_F$ . The physical meaning of such process amounts to removing an electron from the right Fermi point with simultaneous transfer of another electron from the right to the left Fermi point. In interacting electron systems, such processes are possible, but the resulting ‘‘shadow band’’ features tend to be weak.

## IV. GREEN'S FUNCTIONS

A number of important physical properties of one-dimensional electron systems, such as the tunneling density of states and the spectral functions, are expressed in terms of single-electron Green's functions,

$$G_\gamma^+(x,t) = \langle \psi_\gamma(x,t) \psi_\gamma^\dagger(0,0) \rangle, \quad (43a)$$

$$G_\gamma^-(x,t) = \langle \psi_\gamma^\dagger(0,0) \psi_\gamma(x,t) \rangle. \quad (43b)$$

It is well known that in the limit  $|t| \rightarrow \infty$ , the Green's functions show nontrivial power-law behavior, with exponents depending on the interaction strength.<sup>20</sup> This behavior is easily obtained<sup>15</sup> in the bosonization approach based on Eq. (15). In the case of strongly interacting electrons, these results are valid at  $|t| \gg \hbar/J$  and adequately describe the physics of the system at low energies  $|\varepsilon| \ll J \ll E_F$ .

In this paper, we are primarily interested in the regime of intermediate energies,  $J \ll |\varepsilon| \ll E_F$ . To find the Green's functions in this case, instead of Eq. (15), one can use the more general expression (14). We start by transforming electron annihilation operator (14) to a more convenient form.

### A. Annihilation operator for strongly interacting electrons

We first rewrite the lattice operators  $Z_{l,\gamma}$  in Eq. (14) in terms of their Fourier components,

$$Z_{l,\gamma} = \int_{-\pi}^{\pi} \frac{dq}{2\pi} z_{\gamma}(q) e^{iq l}, \quad (44)$$

$$z_{\gamma}(q) = \sum_{l=-\infty}^{\infty} Z_{l,\gamma} e^{-iq l}. \quad (45)$$

Straightforward substitution of Eq. (44) into Eq. (14) yields

$$\begin{aligned} \psi_{\gamma}(x) &= \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} \int_{-\pi}^{\pi} \frac{dq}{2\pi} z_{\gamma}(q) [e^{i(1+q/\pi)[k_F^h x + \phi(x)} \\ &+ e^{i(-1+q/\pi)[k_F^h x + \phi(x)}]. \end{aligned} \quad (46)$$

The two terms in the integrand correspond to removal of right- and left-moving holons, respectively.

Expression (46) can be simplified by noticing that  $z_{\gamma}(q)$  is a  $2\pi$ -periodic function of  $q$ ,

$$\psi_{\gamma}(x) = \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} \int_{-\pi}^{\pi} \frac{dq}{2\pi} z_{\gamma}(q) e^{i(1+q/\pi)[k_F^h x + \phi(x)].} \quad (47)$$

This presentation of the fermion operator is not entirely satisfactory, because the limits of integration here and in Eq. (44) were set rather arbitrarily. Indeed, the argument of the operator  $z_{\gamma}(q)$  is the change of momentum of the spin subsystem when an electron is removed. Since spins in Hamiltonian (4) are attached to lattice sites, the momentum changes by  $q$  or  $q \pm 2\pi$ ,  $q \pm 4\pi$ , etc., are equivalent. This symmetry is lost in bosonized expression (47), but can be restored by extending the limits of  $q$  integration,

$$\psi_{\gamma}(x) = \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} \frac{dq}{2\pi} z_{\gamma}(q) e^{i(1+q/\pi)[k_F^h x + \phi(x)].} \quad (48)$$

The origin of the ambiguity in the definition of the fermion operators can be traced back to the bosonization of the holon operators in Eq. (8b). Indeed, by definition (7), the operator  $l(x)$  has only integer eigenvalues equal to the number of electrons in the region of space from  $-\infty$  to  $x$ , whereas its bosonized expression  $l(x) = \frac{1}{\pi} [k_F^h x + \phi(x)]$  does not explicitly possess this property. To enforce the discreteness of charge in Eq. (14), one can understand  $Z_{l(x),\gamma}$  as

$$Z_{l,\gamma}|_{l=(1/\pi)[k_F^h x + \phi(x)]} \rightarrow \sum_l Z_{l,\gamma} \delta\left(\frac{1}{\pi} [k_F^h x + \phi(x)] - l\right). \quad (49)$$

Then, upon Fourier transformation (44), one recovers Eq. (48).

A similar procedure of bosonization of fermions while preserving the discreteness of their number was suggested by Haldane.<sup>39</sup> Apart from the two terms corresponding to the right- and left-moving fermions, the expression for the fermion operator contains multiparticle contributions with wave vectors near  $\pm 3k_F$ ,  $\pm 5k_F$ , etc. In a typical bosonization calculation, these additional terms give much smaller contributions than the leading ones. Thus, the difference between the results obtained using the two bosonization schemes is smaller than the accuracy of the bosonization approximation and can be ignored. In our case, by using expression (47) instead of Eq. (48), one obtains small spurious features in the spectral function originating from the arbitrarily chosen integration limits. Thus, from now on, we use Eq. (48).

### B. Green's functions at intermediate energies

At small  $J$ , the time evolution of the spin degrees of freedom is very slow, with the typical time scales of order  $\hbar/J$ . Therefore, to find the Green's functions describing physical phenomena at relatively high energies  $|\varepsilon| \gg J$ , one can neglect the time dependence of the correlators of operators  $Z_{l,\gamma}$ . Then, the substitution of Eq. (48) into definitions (43) of the Green's functions yields

$$G_{\gamma}^{\pm}(x, t) = \frac{1}{2\pi\alpha} \int_{-\infty}^{\infty} \frac{dq}{2\pi} c_{\gamma}^{\pm}(q) e^{i(1+q/\pi)k_F^h x} g_q^{\pm}(x, t). \quad (50)$$

Here, the static correlators  $c_{\gamma}^{\pm}(q)$  are defined by

$$c_{\gamma}^{+}(q) = \sum_l \langle Z_{l,\gamma} Z_{0,\gamma}^{\dagger} \rangle e^{-iq l}, \quad (51a)$$

$$c_{\gamma}^{-}(q) = \sum_l \langle Z_{0,\gamma}^{\dagger} Z_{l,\gamma} \rangle e^{-iq l}, \quad (51b)$$

where  $\langle \dots \rangle$  denotes thermal averaging over the equilibrium states of Hamiltonian (4). The time dependence is contained in the correlators  $g_q^{\pm}$  defined in terms of the charge variables,

$$g_q^{+}(x, t) = \langle e^{i[(1+q/\pi)\phi(x,t) - \theta(x,t)]} e^{-i[(1+q/\pi)\phi(0,0) - \theta(0,0)]} \rangle, \quad (52a)$$

$$g_q^{-}(x, t) = \langle e^{-i[(1+q/\pi)\phi(0,0) - \theta(0,0)]} e^{i[(1+q/\pi)\phi(x,t) - \theta(x,t)]} \rangle. \quad (52b)$$

These correlation functions are easily computed using the standard techniques for averaging the exponentials of bosonic fields.<sup>15</sup> In the most important regime  $|t| \ll \hbar/T$ , one finds

$$g_q^{\pm}(x, t) = \left( \frac{\pm i\alpha}{x - v_{\rho} t \pm i\alpha} \right)^{\zeta_q^{\pm}} \left( \frac{\mp i\alpha}{x + v_{\rho} t \mp i\alpha} \right)^{\zeta_q^{\mp}}, \quad (53)$$

where

$$\zeta_q^{\pm} = \frac{1}{4} \left[ \sqrt{K} \left( 1 + \frac{q}{\pi} \right) \pm \frac{1}{\sqrt{K}} \right]^2. \quad (54)$$

The correlators  $c_{\gamma}^{\pm}(q)$  are determined by the properties of Heisenberg spin chain (4). Unlike the holon correlators

$g_q^\pm(x, t)$ , in general,  $c_\gamma^\pm(q)$  cannot be computed using the bosonization approach. We discuss their behavior in detail below.

### C. Static correlators $c_\gamma^\pm(q)$

To find  $c_\gamma^\pm(q)$ , one has to perform the averaging in Eqs. (51) over the eigenstates of Heisenberg Hamiltonian (4). The results crucially depend on the relation between the temperature  $T$  and the exchange constant  $J$ . At  $T \sim J$ , they can only be studied numerically. Some analytical results can be found in the cases of high and low temperatures.

#### 1. General mathematical properties of $c_\gamma^+(q)$ and $c_\gamma^-(q)$

We start by establishing interesting relations between the functions  $c_\gamma^+(q)$  and  $c_\gamma^-(q)$ , which follow from their definition (51) and do not depend on the temperature or the specific form (4) of the Hamiltonian of the spin chain. We first notice that the correlators  $\langle Z_{l,\gamma} Z_{0,\gamma}^\dagger \rangle$  and  $\langle Z_{0,\gamma}^\dagger Z_{l,\gamma} \rangle$  are real and satisfy the following relations:

$$\langle Z_{0,\gamma} Z_{0,\gamma}^\dagger \rangle = 1,$$

$$\langle Z_{l,\gamma} Z_{0,\gamma}^\dagger \rangle = \langle Z_{-l,\gamma} Z_{0,\gamma}^\dagger \rangle,$$

$$\langle Z_{0,\gamma}^\dagger Z_{l,\gamma} \rangle = \langle Z_{0,\gamma}^\dagger Z_{-l,\gamma} \rangle.$$

From definitions (51), it then follows that the correlators  $c_\gamma^\pm(q)$  are real and even functions of  $q$ . Also, definition (51a) of  $c_\gamma^+(q)$  can be rewritten as

$$c_\gamma^+(q) = 1 + 2 \operatorname{Re} \sum_{l=1}^{\infty} \langle Z_{l,\gamma} Z_{0,\gamma}^\dagger \rangle e^{-iq l}. \quad (55)$$

Furthermore, one can establish a simple relation between operators  $Z_{l,\gamma} Z_{0,\gamma}^\dagger$  and  $Z_{0,\gamma}^\dagger Z_{l,\gamma}$ . Both of them add and remove a site with spin  $\gamma$  at different positions on the spin chain. If a site is added first, the numbering of all the subsequent sites is shifted by 1. Thus,  $Z_{l,\gamma} Z_{0,\gamma}^\dagger = Z_{0,\gamma}^\dagger Z_{l-1,\gamma}$ . (Commutation of operators  $Z_{l,\gamma}$  is discussed in more detail in Appendix A.) Then, using Eq. (51b), one obtains

$$\langle Z_{l,\gamma} Z_{0,\gamma}^\dagger \rangle = \int_{-\pi}^{\pi} \frac{dq}{2\pi} c_\gamma^-(q) e^{iq(l-1)}.$$

Substituting this relation into Eq. (55), one finds

$$c_\gamma^+(q) = 1 + c_\gamma^-(q) \cos q + \int_{-\pi}^{\pi} \frac{dq'}{2\pi} \frac{\sin \frac{q'+q}{2}}{\sin \frac{q'-q}{2}} c_\gamma^-(q'). \quad (56)$$

Similarly, one can express  $c_\gamma^-(q)$  in terms of  $c_\gamma^+(q)$ ,

$$c_\gamma^-(q) = c_\gamma^+(q) \cos q + \int_{-\pi}^{\pi} \frac{dq'}{2\pi} \cot \frac{q-q'}{2} \sin q' c_\gamma^+(q'). \quad (57)$$

The integrals over  $q'$  in Eqs. (56) and (57) should be understood as the principal value.

Relations (56) and (57) simplify considerably at  $q=0$  and  $q=\pi$ . In these cases, we find

$$c_\gamma^+(0) - c_\gamma^-(0) = 1 + \langle Z_{0,\gamma}^\dagger Z_{0,\gamma} \rangle, \quad (58a)$$

$$c_\gamma^+(\pi) + c_\gamma^-(\pi) = 1 - \langle Z_{0,\gamma}^\dagger Z_{0,\gamma} \rangle. \quad (58b)$$

The operator  $Z_{0,\gamma}^\dagger Z_{0,\gamma}$  destroys and then recreates a site with a given spin  $\gamma$ . Thus, the average  $\langle Z_{0,\gamma}^\dagger Z_{0,\gamma} \rangle$  is the probability to find spin  $\gamma$  at site  $l=0$ . In the absence of magnetic field, one expects  $\langle Z_{0,\gamma}^\dagger Z_{0,\gamma} \rangle = \frac{1}{2}$ , resulting in

$$c_\gamma^+(0) - c_\gamma^-(0) = \frac{3}{2}, \quad (59a)$$

$$c_\gamma^+(\pi) + c_\gamma^-(\pi) = \frac{1}{2}. \quad (59b)$$

Relations (56) and (57) follow from the definition of the correlators  $c_\gamma^\pm(q)$  and are not sensitive to the specific Hamiltonian (4) of the spin chain. The only property of Hamiltonian (4) important for relations (58) is the spin symmetry. At the same level of universality, one can find the full dependences  $c_\gamma^\pm(q)$  in the high-temperature limit.

#### 2. High temperature $T \gg J$

At high temperatures, correlators similar to  $c_\gamma^\pm(q)$  have been studied by Penc and Serhan.<sup>40</sup> They noticed that at  $J/T \rightarrow 0$ , the spins are completely uncorrelated, and the correlator  $\langle Z_{l,\gamma} Z_{0,\gamma}^\dagger \rangle$  is simply the probability of finding  $|l|$  random spins at sites  $0, 1, \dots, l-1$  (or  $l, l+1, \dots, -1$  for negative  $l$ ) in a given state  $\gamma$ . Thus,  $\langle Z_{l,\gamma} Z_{0,\gamma}^\dagger \rangle = 1/2^{|l|}$  and, similarly,  $\langle Z_{0,\gamma}^\dagger Z_{l,\gamma} \rangle = 1/2^{|l|+1}$ . Substituting these expressions into Eq. (51), one finds

$$c_\gamma^+(q) = 2c_\gamma^-(q) = \frac{3}{5 - 4 \cos q}, \quad T \gg J. \quad (60)$$

One can check explicitly that these results are consistent with relations (56) and (57).

#### 3. Zero temperature

At zero temperature, the correlators  $c_\gamma^+(q)$  and  $c_\gamma^-(q)$  carry nontrivial information about the spin correlations in the ground state of the antiferromagnetic Heisenberg spin chain (4). Although model (4) is exactly solvable, no exact results are known for the correlators  $c_\gamma^\pm(q)$ . The quantity analogous to  $c_\gamma^-(q)$  was first studied by Sorella and Parola<sup>41</sup> who used the results of numerical diagonalization of spin chains of up to 22 sites.<sup>42</sup> Their results indicated that  $c_\gamma^-(q)$  is extremely small at  $0 < q < \pi/2$ , whereas  $c_\gamma^-(q) \sim 1$ . They interpreted this behavior as the effect of ‘‘spinon pseudo-Fermi-surface,’’ which can be rephrased as follows.

Let us consider the ground state of fermionized version (20) of the Heisenberg model. Ignoring interactions (20c) between the fermions, one can picture the ground state as shown in Fig. 2(a). The function  $c_\uparrow^-(q)$  is defined in terms of the correlators  $\langle Z_{0,\uparrow}^\dagger Z_{l,\uparrow} \rangle$ . By removing a spin- $\uparrow$  site, the op-

erator  $Z_{l,\uparrow}$  destroys a fermion. Ignoring for the moment other aspects of  $Z_{l,\uparrow}$ , one concludes that  $c_{\uparrow}^{-}(q)$  should vanish at  $-\pi/2 < q < \pi/2$ , as the fermion states with those values of the wave vector are empty [see Fig. 2(a)]. A similar argument for the correlator  $c_{\uparrow}^{+}(q)$  shows that it should vanish in the region of  $q$  space below the Fermi surface. Noticing that in the absence of magnetic field specific spin direction is unimportant, one concludes that

$$c_{\uparrow}^{+}(q) = 0 \quad \text{at} \quad \frac{\pi}{2} < q < \frac{3\pi}{2}, \quad (61a)$$

$$c_{\uparrow}^{-}(q) = 0 \quad \text{at} \quad -\frac{\pi}{2} < q < \frac{\pi}{2}. \quad (61b)$$

Conditions (61a) and (61b) refer to the values of  $q$  between  $-\pi/2$  and  $3\pi/2$ ; outside of that region, they can be inferred using the  $2\pi$  periodicity of the functions  $c_{\uparrow}^{\pm}(q)$ .

The above picture neglects two important aspects of the problem: (i) the fermions interact with each other, Eq. (20c), and (ii) the operator  $Z_{l,\uparrow}$  not only destroys a fermion but also removes a site from the spin chain. There is no *a priori* reason to expect that the effects of these approximations should be small. However, the bosonization approach to calculation of  $c_{\uparrow}^{\pm}(q)$  near the Fermi points  $q = \pm\pi/2$  (outlined below) shows that although each effect is significant, they mostly compensate each other. Numerically, this compensation results<sup>21,41</sup> in  $c_{\uparrow}^{-}(q) < 0.01$  for  $|q| < \pi/2$ . Based on their numerical results, Sorella and Parola<sup>41</sup> conjectured that in an infinitely long spin chain,  $c_{\uparrow}^{-}(q) = 0$  at  $|q| < \pi/2$ . From this point of view, the numerically small values of  $c_{\uparrow}^{-}(q)$  in this interval should be viewed as a finite-size effect.

A more detailed study of the static correlators of spin chain (4) was performed by Penc *et al.*<sup>21</sup> They studied both correlators  $c_{\uparrow}^{+}(q)$  and  $c_{\uparrow}^{-}(q)$  (in slightly different notations) and found that not only  $c_{\uparrow}^{-}(q)$  is numerically small at  $0 < q < \pi/2$  but also  $c_{\uparrow}^{+}(q)$  is small at  $\pi/2 < q < \pi$ . Although these observations are consistent with the single-particle Fermi surface prediction [Eqs. (61a) and (61b)], no significant size effect was observed. Penc *et al.*<sup>21</sup> concluded that contrary to Eqs. (61a) and (61b), correlators  $c_{\uparrow}^{+}(q)$  and  $c_{\uparrow}^{-}(q)$  do not vanish exactly in the “forbidden” regions of the wave vector  $q$ .

We now show analytically that although conditions (61a) and (61b) may be a good approximation of actual behavior of correlators  $c_{\uparrow}^{\pm}(q)$ , they contradict to the spin-rotation symmetry of the problem. We showed in Sec. IV C 1 that the correlators  $c_{\uparrow}^{+}(q)$  and  $c_{\uparrow}^{-}(q)$  are related to each other, and if one of them is known on the interval of  $q$  of length  $2\pi$ , the other can be obtained using relations (56) and (57). Conditions (61a) and (61b) define  $c_{\uparrow}^{+}(q)$  over half the period  $2\pi$  and  $c_{\uparrow}^{-}(q)$  over the other half. This is sufficient to uniquely define both functions at the remaining regions of  $q$  space. To accomplish that, one can solve integral equation (57) with conditions (61a) and (61b). This mathematical problem belongs to the class of singular integral equations and can be solved by applying the well-known techniques<sup>43</sup> (see Appendix C). The solution has the form

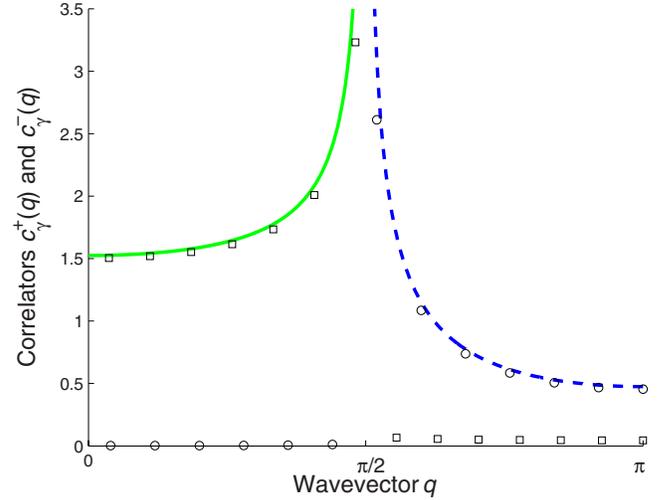


FIG. 3. (Color online) The solution of integral equation (56) with conditions (61a) and (61b). The solid and dashed lines show the behavior of  $c_{\uparrow}^{+}(q)$  and  $c_{\uparrow}^{-}(q)$  described by Eqs. (62a) and (62b), respectively. Squares and circles represent the results of numerical solution (Refs. 21 and 44) for the 26-site chain [Eq. (4)].

$$c_{\uparrow}^{+}(q) = \frac{c_0}{\sqrt{\cos q}} \exp\left(-\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \ln \left| \sin \frac{q-q'}{2} \right| dq'\right) \quad (62a)$$

for  $-\pi/2 < q < \pi/2$  and

$$c_{\uparrow}^{-}(q) = \frac{c_0}{\sqrt{|\cos q|}} \exp\left(-\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \ln \sin \frac{q-q'}{2} dq'\right) \quad (62b)$$

for  $\pi/2 < q < 3\pi/2$ . The normalization constant  $c_0$  is found using Eq. (58),

$$c_0 = \frac{1}{2 \cosh \frac{2G}{\pi}}, \quad (63)$$

where  $G \approx 0.91597$  is Catalan’s constant. Solution (62a), (62b) is plotted in Fig. 3.

As expected, our solution gives an excellent approximation to the numerical data of Ref. 21. Nevertheless, the results in Eqs. (61a), (61b), (62a), and (62b) are not exact. Indeed, from Eqs. (62a) and (62b), we find

$$c_{\uparrow}^{+}(0) = \frac{2}{1 + e^{-4G/\pi}} \approx 1.525, \quad (64a)$$

$$c_{\uparrow}^{-}(\pi) = \frac{2}{e^{4G/\pi} + 1} \approx 0.475. \quad (64b)$$

Taking into consideration that  $c_{\uparrow}^{+}(\pi) = c_{\uparrow}^{-}(0) = 0$ , Eq. (61a) and (61b), we find that spin-symmetry conditions (59) are not satisfied exactly.

To gain better insight into the properties of the correlators  $c_{\gamma}^{\pm}(q)$ , let us consider the case when the wave vector approaches the Fermi point,  $q \rightarrow \pi/2$ . In this regime, one can apply the bosonization approach of Sec. III B. Using expressions (41) for the operators  $Z_{l,\gamma}$  and applying standard techniques<sup>15</sup> for averaging the exponentials of bosonic field over the ground state of Hamiltonian (28a), we find

$$\langle Z_{l,\gamma} Z_{0,\gamma}^{\dagger} \rangle = \left[ e^{i(\pi/2)l} \left( \frac{\tilde{\alpha}}{\tilde{\alpha} + il} \right)^{1/2} + e^{-i(\pi/2)l} \left( \frac{\tilde{\alpha}}{\tilde{\alpha} - il} \right)^{1/2} \right] \times \left( \frac{\tilde{\alpha}^2}{\tilde{\alpha}^2 + l^2} \right)^{(\mathcal{K} + 1/4\mathcal{K} - 1)/4}. \quad (65)$$

In the limit of low energies, the Luttinger-liquid constant  $\mathcal{K} \rightarrow 1/2$ . Using this value, one can find the correlator  $c^{\pm}(q)$  near the Fermi point  $q = \pi/2$  by substituting the component of the Green's function (65) given by the first term in the brackets into Eq. (51a) and replacing the sum over  $l$  with an integral:

$$c_{\gamma}^{\pm}(q) = \int_{-\infty}^{\infty} \left( \frac{\tilde{\alpha}}{\tilde{\alpha} + il} \right)^{1/2} e^{-i(q - \pi/2)l} dl. \quad (66)$$

It is important to note that the integrand is analytic in the lower complex half-plane. Thus,  $c_{\gamma}^{\pm}(q) = 0$  for  $q > \pi/2$ . This conclusion agrees with the prediction of the spinon Fermi surface approximation (61). At  $q < \pi/2$ , the integration in Eq. (66) is straightforward, and we find

$$c_{\gamma}^{\pm}(q) = \chi \frac{\Theta\left(\frac{\pi}{2} - q\right)}{\sqrt{\frac{\pi}{2} - q}}, \quad (67a)$$

where  $\Theta(x)$  is the unit step function, and the value of the numerical coefficient  $\chi \sim 1$  cannot be determined within the bosonization approach. A similar calculation for  $c_{\gamma}^{-}(q)$  results in

$$c_{\gamma}^{-}(q) = \chi \frac{\Theta\left(q - \frac{\pi}{2}\right)}{\sqrt{q - \frac{\pi}{2}}}, \quad (67b)$$

also in agreement with prediction (61) of the spinon Fermi surface picture.

Inverse-square-root singularities (67a) and (67b) of the correlators  $c_{\gamma}^{\pm}(q)$  at the Fermi point are consistent with the numerical data of Ref. 21 and our expressions (62a) and (62b) (see also Fig. 3). In particular, the asymptotes of expressions (62a) and (62b) at  $q \approx \pi/2$  are given by Eq. (67a) and (67b) with  $\chi \approx 0.85$ .

It is important to note that results (67a) and (67b) of the bosonization treatment of the static correlators  $c_{\gamma}^{\pm}(q)$  do not prove the validity of the spinon Fermi surface picture. Indeed, bosonized expressions (41) for operators  $Z_{l,\gamma}$  are only applicable asymptotically near the Fermi point,  $q \rightarrow \pm\pi/2$ . Consequently, the presence of the step function  $\Theta$  in results (67a) and (67b) should be interpreted as

$$\lim_{\delta \rightarrow +0} \frac{c_{\gamma}^{\pm}\left(\frac{\pi}{2} + \delta\right)}{c_{\gamma}^{\pm}\left(\frac{\pi}{2} - \delta\right)} = \lim_{\delta \rightarrow +0} \frac{c_{\gamma}^{-}\left(\frac{\pi}{2} - \delta\right)}{c_{\gamma}^{-}\left(\frac{\pi}{2} + \delta\right)} = 0. \quad (68)$$

Let us now show that the correlator  $c_{\gamma}^{\pm}(q)$  not only does not vanish at  $q > \pi/2$  but, in fact, diverges at  $q \rightarrow \pi/2 + 0$ .

We start by noticing that the conclusion  $c_{\gamma}^{\pm}(q) = 0$  at  $q > \pi/2$  holds only for  $\mathcal{K} = 1/2$ . Indeed, at  $\mathcal{K} \neq 1/2$ , the last factor in expression (65) is no longer unity. More importantly, it is no longer analytic in the lower complex half-plane, leading to the nonvanishing value of integral (66) at  $q > \pi/2$ .

The low-energy properties of Heisenberg spin chain (4) are adequately described by sine-Gordon Hamiltonian (28a). The parameters  $\mathcal{K}$  and  $g$  renormalize at low energies or, equivalently, long length scales  $L$  as

$$\mathcal{K} = \frac{1}{2} + \frac{y}{4}, \quad (69)$$

$$g = \pi \tilde{v} y, \quad (70)$$

$$y = \frac{1}{\ln L}. \quad (71)$$

Thus, at low energies,  $\mathcal{K}$  deviates slightly from  $1/2$ . This deviation gives a correction to the correlator (66) which can be obtained by expansion of Eq. (65) in powers of  $y = 4(\mathcal{K} - 1/2)$ ,

$$\delta c_{\gamma}^{\pm}(q) = -\frac{y^2}{32} \int_{-\infty}^{\infty} \left( \frac{\tilde{\alpha}}{\tilde{\alpha} + il} \right)^{1/2} \ln \frac{\tilde{\alpha}^2 + l^2}{\tilde{\alpha}^2} e^{-i(q - \pi/2)l} dl. \quad (72)$$

At  $q < \pi/2$ , the correction (72) is small compared to leading term (67a) and can be ignored. However, unlike Eq. (67a), correction (72) does not vanish at  $q > \pi/2$ ,

$$\delta c_{\gamma}^{\pm}(q) = \frac{\pi}{32} y^2 \frac{\chi}{\sqrt{q - \frac{\pi}{2}}}, \quad (73)$$

where  $\chi$  is the same numerical coefficient as in Eq. (67a) and (67b).

The deviation of the Luttinger-liquid parameter  $\mathcal{K}$  from the limiting value of  $1/2$  is not the only source of corrections to  $c_{\gamma}^{\pm}(q)$ . Additional corrections originate from the small sine-Gordon term (28b). One can account for this term to second-order perturbation theory using the standard techniques.<sup>15</sup> The resulting correction is factor of 2 greater than Eq. (73). [Apart from direct calculation, this can be shown to follow from the spin-rotation symmetry of the problem (see Appendix D).] We therefore conclude that at  $q$  slightly above the Fermi point  $\pi/2$ , the correlator  $c_{\gamma}^{\pm}$  is given by

$$c_{\gamma}^{+}(q) = \frac{3\pi}{32} y^2 \frac{\chi}{\sqrt{q - \frac{\pi}{2}}}, \quad q \rightarrow \frac{\pi}{2} + 0. \quad (74)$$

The above calculation was performed to second-order perturbation theory in coupling constant  $y$  and did not account for its scaling (71). Since the scaling is logarithmic, and thus slow compared to the leading power-law behavior in Eq. (74), one can simply substitute expression (71) into Eq. (74), choosing the proper value of the length scale  $L \sim 1/(q - \pi/2)$ . We therefore conclude that near the Fermi point  $q = \pi/2$ , the correlator  $c_{\gamma}^{+}(q)$  behaves as

$$c_{\gamma}^{+}(q) = \begin{cases} \frac{\chi}{\sqrt{\frac{\pi}{2} - q}}, & q \rightarrow \frac{\pi}{2} - 0 \\ \frac{3\pi}{32} \frac{1}{\ln^2\left(q - \frac{\pi}{2}\right)} \frac{\chi}{\sqrt{q - \frac{\pi}{2}}}, & q \rightarrow \frac{\pi}{2} + 0. \end{cases} \quad (75a)$$

Analogous calculation for the correlator  $c_{\gamma}^{-}$  gives

$$c_{\gamma}^{-}(q) = \begin{cases} \frac{\chi}{\sqrt{\frac{q - \pi}{2}}}, & q \rightarrow \frac{\pi}{2} + 0 \\ \frac{3\pi}{32} \frac{1}{\ln^2\left(\frac{\pi}{2} - q\right)} \frac{\chi}{\sqrt{\frac{\pi}{2} - q}}, & q \rightarrow \frac{\pi}{2} - 0. \end{cases} \quad (75b)$$

Our results (75a) and (75b) reaffirm our earlier observation that the correlators  $c_{\gamma}^{\pm}(q)$  do not vanish exactly on one side of the Fermi point, as expected from the spinon Fermi surface picture, cf. Eq. (61a) and (61b). Instead, the correlators diverge at  $q \rightarrow \pi/2$ , albeit slower than on the “main” side of the Fermi point. The numerical data of Ref. 21 shown in Fig. 3 indicate that the small values of  $c_{\gamma}^{+}(q)$  at  $q > \pi/2$  do increase near the Fermi point. However, studies of much longer systems are needed to verify asymptotes (75a) and (75b).

## V. SPECTRAL FUNCTIONS

The spectral functions  $A_{\gamma}^{+}(k, \omega)$  and  $A_{\gamma}^{-}(k, \omega)$  of a one-dimensional electron system are defined as Fourier transforms

$$A_{\gamma}^{\pm}(k, \omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-ikx + i\omega t} G_{\gamma}^{\pm}(x, t) \quad (76)$$

of the Green’s functions (43). The components  $A_{\gamma}^{+}(k, \omega)$  and  $A_{\gamma}^{-}(k, \omega)$  characterize the particle and hole parts of the excitation spectrum, respectively. In this section, we study the behavior of the spectral functions at frequencies  $\omega$  in the range

$$J, T \ll \hbar\omega \ll D_{\rho}. \quad (77)$$

Under these restrictions, one can express the Green’s functions in forms (50), (51), and (53). Performing the integration with respect to  $x$  and  $t$ , one finds

$$A_{\gamma}^{\pm}(k, \omega) = \int_{q_k - q_{\omega}}^{q_k + q_{\omega}} dq \frac{\Theta(\pm\omega)}{2v_{\rho} k_F^h} \frac{(\alpha k_F^h / 2\pi)^{\zeta_q^+ \zeta_q^- - 1}}{\Gamma(\zeta_q^+) \Gamma(\zeta_q^-)} \times \frac{c_{\gamma}^{\pm}(q)}{[q - q_k + q_{\omega}]^{1 - \zeta_q^+} [q_k + q_{\omega} - q]^{1 - \zeta_q^-}}, \quad (78)$$

where

$$q_k = \frac{\pi}{k_F^h} (k - k_F^h), \quad q_{\omega} = \frac{\pi|\omega|}{v_{\rho} k_F^h}. \quad (79)$$

According to Eq. (78), at small frequencies  $\omega \ll v_{\rho} k_F^h$ , the spin wave vector  $q$  must be close to  $q_k$ . This can be understood in terms of the energy and momentum conservation laws. Since the energy  $\omega$  is small, the holon component of the electron must have momentum near  $\pm k_F^h$ . If the electron momentum  $k$  is near one of those values, the spinons carry no momentum, i.e.,  $q=0$  (or  $2\pi$ ), in agreement with  $q \approx q_k \approx 0$ . On the other hand, if  $k$  is not near  $\pm k_F^h$ , the difference of the momenta  $k - k_F^h = nq_k$  is transferred into the spin subsystem. (Note that the electron density  $n = k_F^h / \pi$ .)

### A. Zero momentum peak

To find the momentum dependence of spectral functions (78) at low frequencies, one can replace  $q \rightarrow q_k$  in  $c_{\gamma}^{\pm}(q)$  and  $\zeta_q^{\pm}$ . The remaining integration is straightforward and gives

$$A_{\gamma}^{\pm}(k, \omega) = \frac{\Theta(\pm\omega) c_{\gamma}^{\pm}(q_k)}{2v_{\rho} k_F^h \Gamma(\zeta(k))} \left( \frac{D_{\rho}}{\hbar|\omega|} \right)^{1 - \zeta(k)}, \quad (80)$$

where

$$\zeta(k) = \zeta_{q_k}^+ + \zeta_{q_k}^- = \frac{K}{2} \left( \frac{k}{k_F^h} \right)^2 + \frac{1}{2K}. \quad (81)$$

In the low-frequency limit  $\hbar|\omega|/D_{\rho} \rightarrow 0$ , the spectral function  $A_{\gamma}^{\pm}$  has a sharp Gaussian peak as a function of the electron momentum centered at  $k=0$ . In the spin-incoherent limit  $J/T \rightarrow 0$ , a similar Gaussian peak in the momentum dependence of the spectral function was found by Fiete *et al.*<sup>27</sup> Our expression (80) is valid at arbitrary  $J/T$ , with the temperature dependence entering Eq. (80) via the functions  $c_{\gamma}^{\pm}(q)$ . In particular, using the numerical results<sup>21</sup> for  $c_{\gamma}^{\pm}(q)$ , we can access the zero-temperature limit.

Peak (80) gives the leading contribution to the density of states at low energies,

$$\begin{aligned}
 v_{\gamma}^{\pm}(\varepsilon) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi\hbar} A_{\gamma}^{\pm}(k, \varepsilon/\hbar) \\
 &= \frac{\Theta(\pm\varepsilon)}{\pi\hbar v_{\rho}} \sqrt{\frac{\pi}{8K}} \frac{c_{\gamma}^{\pm}(\pi)}{\Gamma\left(\frac{1}{2K}\right)} \left(\frac{D_{\rho}}{|\varepsilon|}\right)^{1-1/2K} \frac{1}{\sqrt{\ln \frac{D_{\rho}}{|\varepsilon|}}}.
 \end{aligned} \tag{82}$$

This result reproduces the expressions for the tunneling density of states at  $\varepsilon/D_{\rho} \rightarrow 0$  reported in Ref. 32.

It is important to note that the Gaussian peak in spectral function (80) cannot be obtained within the standard Luttinger-liquid theory of spectral functions,<sup>46,47</sup> which applies only near  $k = \pm k_F$ . Peak (80) is due to the holon states with wave vectors near  $k_F^h = 2k_F$  and the spin excitations with  $q$  near  $\pi$ , well below the spinon Fermi surface (see Fig. 2).

An interesting consequence of the fact that the Gaussian peak is dominated by spin excitations away from the Fermi points is the dramatic difference of the peak heights for the spectral functions  $A_{\gamma}^{+}$  and  $A_{\gamma}^{-}$ . Indeed, as we discussed in Sec. IV C, at zero temperature,  $c_{\gamma}^{+}(\pi) \ll c_{\gamma}^{-}(\pi)$ , whereas at  $T \gg J$ , one has  $c_{\gamma}^{+} = 2c_{\gamma}^{-}$ . Thus, the heights of the peaks at  $k=0$  strongly depend on temperature, and the peak in  $A_{\gamma}^{-}$  is much more pronounced than that in  $A_{\gamma}^{+}$  at  $T \ll J$ .

In our derivation of expression (80) for the spectral functions, we replaced  $q \rightarrow q_k$  in the arguments of the spin correlators  $c_{\gamma}^{\pm}(q)$  in Eq. (78). This procedure is well justified if  $q_k$  is not too close to points  $\pi(2s-1)/2$  (with  $s=0, \pm 1, \pm 2, \dots$ ), where at zero temperature the correlators  $c_{\gamma}^{\pm}(q)$  have sharp inverse-square-root singularities (see Fig. 3). In addition, one can still use Eq. (80) if the singularities of  $c_{\gamma}^{\pm}(q)$  are smeared by finite temperature by  $\delta q \sim T/J \gg q_{\omega}$ . However, at very low temperatures,

$$T \lesssim \frac{\hbar\omega J}{D_{\rho}}, \tag{83}$$

the approximation leading to Eq. (80) fails near  $q_k = \pi(2s-1)/2$ . Thus, at low temperatures, the spectral functions have nontrivial behavior in the vicinity of  $k = (2s+1)k_F$ . Below, we consider the zero-temperature behavior of the spectral functions near these points.

## B. Fermi surface features

When the electron wave vector approaches the Fermi point  $k_F$ , we have  $q_k \rightarrow -\pi/2$ . From Eq. (67a) and the symmetry  $c_{\gamma}^{+}(q) = c_{\gamma}^{+}(-q)$ , we conclude that near  $q = -\pi/2$ , one can approximate  $c_{\gamma}^{+}(q)$  as

$$c_{\gamma}^{+}(q) = \chi \frac{\Theta\left(\frac{\pi}{2} + q\right)}{\sqrt{\frac{\pi}{2} + q}}. \tag{84}$$

To explore the fine structure of the Fermi surface features at zero temperature, instead of the approximate expression (80), one should use the more general formula (78). Then, instead

of one singularity at  $q_k = -\pi/2$ , one finds two singularities at  $q_k = -\pi/2 \pm q_{\omega}$ . Given definitions (79), one concludes that the spectral function  $A_{\gamma}^{+}$  has singularities at two points  $k = k_F \pm \omega/v_{\rho}$ , above and below  $k_F$ .

In evaluating the singular terms in spectral function (78), one can approximate the exponents  $\zeta_q^{\pm}$  by their values at  $q = -\pi/2$ . Using expressions (54), we then find  $\zeta_{-\pi/2}^{-} = \zeta_{-\pi/2}^{+} - 1/2 = \zeta_0$ , where

$$\zeta_0 = \frac{(2-K)^2}{16K} = \frac{(1-K_{\rho})^2}{8K_{\rho}}. \tag{85}$$

The stronger of the two singularities is the one above  $k_F$ , at  $k = k_F + \omega/v_{\rho}$ . It appears when inverse-square-root singularity (84) in  $c_{\gamma}^{+}(q)$  is near the lower limit of integral (78). The most singular term in the spectral function  $A_{\gamma}^{+}$  can then be found by extending the upper limit to infinity, resulting in

$$\begin{aligned}
 A_{\gamma}^{+}(k, \omega) &\simeq \frac{\chi(\alpha/2v_{\rho})^{2\zeta_0-1/2} B}{2\pi\Gamma(\zeta_0 + \frac{1}{2})\Gamma(\zeta_0)} \sqrt{\frac{\pi}{2v_{\rho}k_F}} \\
 &\quad \times [2\omega|\omega - v_{\rho}(k - k_F)|]^{\zeta_0-1/2},
 \end{aligned} \tag{86}$$

where the constant  $B$  is defined in terms of the beta function,

$$B = \begin{cases} B(\frac{1}{2} - \zeta_0, \frac{1}{2}), & \omega - v_{\rho}(k - k_F) \rightarrow +0 \\ B(\frac{1}{2} - \zeta_0, \zeta_0), & \omega - v_{\rho}(k - k_F) \rightarrow -0. \end{cases} \tag{87}$$

Power-law singularity (86) of the spectral function  $A_{\gamma}^{+}$  at  $k = k_F + \omega/v_{\rho}$  with the exponent  $\zeta_0 - 1/2$  is consistent with the results of the Luttinger-liquid theory.<sup>46,47</sup>

It is worth noting that at

$$K = 2K_{\rho} > 6 - 4\sqrt{2} \approx 0.343, \tag{88}$$

spectral function (86) diverges at  $k = k_F + \omega/v_{\rho}$ . Nevertheless, the dominant contribution to density of states (82) at low energies is given by Gaussian peak (80) at  $k=0$ .

The second singularity of the spectral function  $A_{\gamma}^{+}$  is below  $k_F$ , at  $k = k_F - \omega/v_{\rho}$ . It emerges when singularity (84) of  $c_{\gamma}^{+}(q)$  is at the upper limit of integral (78). As  $k$  approaches  $k_F - \omega/v_{\rho}$  from above, the width of the integration region shrinks to zero, and at  $k < k_F - \omega/v_{\rho}$ , the integral vanishes. The singularity at  $k \rightarrow k_F - \omega/v_{\rho}$  has the form

$$\begin{aligned}
 A_{\gamma}^{+}(k, \omega) &\simeq \frac{\Theta(k - (k_F - \omega/v_{\rho}))\chi\zeta_0(\alpha/2v_{\rho})^{2\zeta_0-1/2}}{2\pi[\Gamma(\zeta_0 + \frac{1}{2})]^2\sqrt{2v_{\rho}k_F}} \\
 &\quad \times (2\omega)^{\zeta_0-1} [\omega + v_{\rho}(k - k_F)]^{\zeta_0}.
 \end{aligned} \tag{89}$$

Unlike feature (86) above  $k_F$ , this term always vanishes at the singularity because  $\zeta_0 > 0$ . The power-law feature in the spectral function with the exponent  $\zeta_0$  was obtained earlier<sup>46,47</sup> in the framework of the Luttinger-liquid theory.

The fact that power-law feature (89) appears only on one side of the point  $k = k_F - \omega/v_{\rho}$  is a consequence of the ‘‘spinon Fermi surface’’ approximation. In a more careful treatment, the correlator  $c_{\gamma}^{+}(q)$  does not vanish at  $q < -\pi/2$ , but instead has an inverse-square-root singularity with an additional factor  $1/\ln^2(q + \pi/2)$  [cf. Eq. (75a)]. Thus, cusp (89) should appear on both sides of the point  $k = k_F - \omega/v_{\rho}$ , albeit with an additional logarithmic suppression factor at  $k < k_F - \omega/v_{\rho}$ . In

the Luttinger-liquid theory, this feature would be caused by the marginally irrelevant perturbation  $\cos(2\sqrt{2}\phi_\sigma)$  that was not accounted for in Refs. 46 and 47.

At  $\omega < 0$ , the spectral function  $A_\gamma^-$  has similar power-law singularities at  $k = k_F \pm \omega/v_\rho$ . The respective expressions can be obtained by replacing  $\omega \rightarrow -\omega$  and  $k - k_F \rightarrow k_F - k$  in Eqs. (86) and (89).

### C. Shadow band features

The singularities in the spectral functions emerge when excitations in both charge and spin subsystems are near the Fermi points  $k_F^h$  and  $(2s-1)\pi/2$ , respectively. In particular, if the holon momentum is  $k_F^h = 2k_F$  and the spinon momentum  $nq = (2k_F/\pi)(\pi/2) = k_F$ , the electron momentum is  $3k_F$ . Formally, the features in the spectral functions are caused by singularities (67a) and (67b) appearing inside the narrow integration region in Eq. (78).

Similar to the singularities near  $k_F$ , one expects to find two features, at  $k = 3k_F \pm \omega/v_\rho$ , when either of the limits of integration crosses the point  $q = \pi/2$ . Since the features in the spectral functions near  $3k_F$  are weaker than the ones near  $k_F$ , we discuss only the stronger of the two singularities in  $A_\gamma^+$ . It appears when the lower limit  $q_k - q_\omega$  is near  $\pi/2$  and corresponds to  $k = 3k_F + \omega/v_\rho$ .

The behavior of the spectral function is controlled by exponents (54) at  $q = \pi/2$ , which can be expressed in terms of a single parameter

$$\zeta_1 = \frac{(3K-2)^2}{16K} = \frac{(3K_\rho-1)^2}{8K_\rho} \quad (90)$$

as  $\zeta_q^- = \zeta_q^+ - 3/2 = \zeta_1$ . Then, the singular term in the spectral function takes the form

$$A_\gamma^+(k, \omega) \simeq \frac{\Theta(3k_F + \omega/v_\rho - k) \chi(\alpha/2v_\rho)^{2\zeta_1+1/2}}{2\pi(\zeta_1 + \frac{1}{2})[\Gamma(\zeta_1)]^2 \sqrt{2v_\rho k_F}} \times (2\omega)^{\zeta_1+1/2} [\omega - v_\rho(k - 3k_F)]^{\zeta_1-1/2}. \quad (91)$$

Similar to singularity (89), one will find a weaker feature on the other side of the singularity, i.e., at  $k \rightarrow (3k_F + \omega/v_\rho) + 0$ , if instead of Eq. (67a) a more accurate approximation [Eq. (75a)] for the correlator  $c_\gamma^+(q)$  is applied.

The features in the spectral functions at  $k = 3k_F + \omega/v_\rho$  have been observed in numerical data for the infinite- $U$  Hubbard model by Penc *et al.*,<sup>22</sup> who identified it with the so-called shadow band.<sup>48</sup> Our formula (91) provides analytic expression for the spectral function at the shadow band position in the limit of low frequencies  $\omega \ll D_\rho/\hbar$ . Unlike the numerical treatment of Ref. 22, our result is not limited to the Hubbard model with only on-site repulsion and can be applied to systems with any interaction range.

In addition to the shadow band feature near  $k = 3k_F$ , the periodicity of the correlators  $c_\gamma^\pm(q)$  results in singularities of the spectral functions at all odd multiples of  $k_F$ . Similar to the features near  $k_F$  and  $3k_F$ , one finds a pair of singularities at  $k = (2s+1)k_F \pm \omega/v_\rho$  for  $s = 2, 3, \dots$ . The stronger singularity in each pair is the one at  $k = (2s+1)k_F + \omega/v_\rho$ , where one finds

$$A_\gamma^\pm(k, \omega) \propto |\omega|^{\zeta_s+s-1/2} \{\omega - v_\rho[k - (2s+1)k_F]\}^{\zeta_s-1/2}, \quad (92)$$

with

$$\zeta_s = \frac{[(2s+1)K-2]^2}{16K} = \frac{[(2s+1)K_\rho-1]^2}{8K_\rho}. \quad (93)$$

At  $s=0, 1$ , this expression is consistent with our earlier results (86) and (91). For a given  $s$ , the strongest (inverse-square-root) divergence of the spectral functions at  $k = (2s+1)k_F + \omega/v_\rho$  is achieved when  $\zeta_s = 0$ , i.e.,  $K_\rho = 1/(2s+1)$ . Since in the case of strong repulsion  $K_\rho < 1/2$ , this condition cannot be satisfied for  $s=0$ ; the lowest possible value of  $\zeta_0$  is  $1/16$ .

## VI. SUMMARY AND DISCUSSION OF THE RESULTS

In this paper, we have developed the theory of one-dimensional electron systems in the regime of very strong interactions. This regime emerges when the repulsion between electrons strongly suppresses exchange of their spins,  $J \ll E_F$ . Our theory is based on the Hamiltonian  $H = H_\rho + H_\sigma$ , with charge part (2) brought to form (9) by means of conventional bosonization, while the spin contribution  $H_\sigma$  is the Hamiltonian of Heisenberg spin chain (4). The most important ingredient of the theory is expression (8) for the electron creation and annihilation operators in terms of the charge and spin degrees of freedom.

In our technique, the charge excitations are bosonized, and thus the applicability of the results is limited to energies well below  $E_F$ . On the other hand, the spin excitations are treated more carefully, so we can access the energy scales both below and above  $J$ . At energies below  $J$ , the standard approach based on bosonization procedure (15) and Hamiltonian (16) can be applied. We showed in Sec. III that at  $\varepsilon \ll J$ , our expressions (8) for the electron operators reproduce bosonization formulas (15). The advantage of our method is that unlike bosonization procedure (15), it can also be applied at energy scales  $\varepsilon \gtrsim J$ .

The main difficulty in applying our technique is the need to find the Green's functions of operators  $Z_{l,\gamma}$  with Heisenberg Hamiltonian (4). The problem is simplified for the most interesting case  $\varepsilon \gg J$ , when the slow time dependence of the spin degrees of freedom can be ignored. In this case, the single-particle Green's functions of electrons can be expressed in terms of the static spin correlators  $c_\gamma^\pm(q)$ , which were studied in Refs. 21, 23, 40, and 41. Additional useful properties of these correlators are derived in Sec. IV C.

We have applied our technique to the calculation of the spectral functions of strongly interacting one-dimensional electron systems in Sec. V. Experimentally, the spectral functions can, in principle, be studied by angle-resolved photoemission spectroscopy. However, we are not aware of such experiments on one-dimensional conductors in the regime of strong interactions. A more promising approach is to observe momentum-resolved tunneling between two parallel quantum wires. Such measurements have been recently reported by Auslaender *et al.*<sup>49,50</sup> The regime of strong interactions can be achieved in a quantum wire by reducing the electron density. Unfortunately, at low densities, the effects of disor-

der are also amplified, and the tunneling into states of given momentum is no longer possible.<sup>50</sup> Thus, to observe the predicted features in the spectral functions one would have to manufacture quantum wires with even less disorder than in Refs. 49 and 50.

Our results can be compared with previous theoretical studies of the spectral functions of one-dimensional electron systems. In particular, Meden and Schönhammer<sup>46</sup> and Voit<sup>47</sup> studied the spectral functions in the framework of the bosonization approach. In the case of a strongly interacting system, their results are valid only at energies below  $J$ . However, they can still be compared with our results at  $\omega \gg J/\hbar$  by taking the formal limit  $v_\sigma \rightarrow 0$  in the bosonization results, as suggested by Penc *et al.*<sup>21</sup> The spectral functions of Refs. 46 and 47 show singularities at  $k = k_F \pm \omega/v_\rho$ , which are not sensitive to the spin velocity  $v_\sigma$ . As expected, the power-law behavior of the spectral functions at those singularities is in agreement with our results (86) and (89). Similarly, in the  $v_\sigma \rightarrow 0$  limit, the singularities at  $\omega = \pm v_\sigma(k - k_F)$  of the spectral functions of Refs. 46 and 47 show the same power-law frequency dependence as Gaussian peak (80) in the tail region (at  $k = k_F$ ).

Since the bosonization technique accounts only for the electrons near the Fermi points, our results for  $k$  away from  $\pm k_F$  cannot be compared with those of Refs. 46 and 47. In particular, the main contribution to the tunneling density of states at low energies is due to the peak in the spectral functions centered at  $k=0$  [Eq. (80)]. Consequently, our expression (82) for the density of states in the case of the Hubbard model ( $K=1$ ) gives a larger result  $\nu \propto \varepsilon^{-1/2}$  than the Fermi-surface contribution  $\nu \propto \varepsilon^{-3/8}$  by Penc *et al.*<sup>21</sup> The inverse-square-root dependence of the density of states on energy was obtained earlier<sup>24–26</sup> in the case of  $T \gg J$ . Our results show<sup>32</sup> that the same dependence also holds for  $T \lesssim J$ . Physically, the enhancement of the density of states at low energy is analogous to that in the x-ray absorption edge problem,<sup>51</sup> with the spin excitations creating the effective core-hole potential for the holons.<sup>22,32</sup>

A detailed comparison can be made between our results for the spectral function and those of Penc *et al.*<sup>22</sup> The latter work studied numerically the quarter-filled Hubbard model in the limit of infinite on-site repulsion  $U$ . To compare their results with ours, one should assume  $K=1$ , and consequently  $\zeta_0 = \zeta_1 = 1/16$ . Our results (86) and (91) indicate that power-law peak with exponents  $\zeta_{0,1} - 1/2 = -7/16$  should appear at  $k = k_F + \omega/v_\rho$  and  $k = 3k_F + \omega/v_\rho$ . The data of Ref. 22 do show singularities at those lines in the  $(k, \omega)$  plane. In addition, as  $\omega \rightarrow 0$ , the singularity at  $k = k_F + \omega/v_\rho$  is expected to grow as  $\omega^{-7/16}$ , whereas the one at  $k = 3k_F + \omega/v_\rho$  is expected to be suppressed as  $\omega^{9/16}$ . The data of Ref. 22 do show this qualitative behavior. Finally, the data of Ref. 22 clearly show a weak feature at  $k = k_F - \omega/v_\rho$ , which becomes more prominent at  $\omega \rightarrow 0$ . This feature is consistent with our result (89) which at  $\zeta_0 = 1/16$  behaves as  $\omega^{-15/16} [k - (k_F - \omega/v_\rho)]^{1/16}$ .

One should note that the numerical data of Ref. 22 do not show a peak at  $k=0$  that we expect based on Eq. (80). At  $\omega \rightarrow +0$ , the spectral function appears to be very small. This can be understood as a result of the smallness of  $c_\gamma^+(\pi) \approx 0.044$ . At  $\omega \rightarrow -0$  the spectral function<sup>22</sup> is not small, but

instead of a Gaussian peak, it shows a rather flat minimum at  $k=0$ . This can be understood by noticing that the  $k$ -dependent prefactor  $c_\gamma^-(q_k)/\Gamma(\zeta(k))$  in Eq. (80) has a minimum at  $k=0$ . In the limit  $|\omega|/D_\rho \rightarrow 0$ , the last factor in Eq. (80) dominates, and we find a peak. However, in a finite system, one cannot access the values of  $\hbar|\omega|$  below the level spacing. Substituting the parameters of the Hubbard chain used in Ref. 22 into Eq. (80) and using approximation (62b) for  $c_\gamma^-(q)$ , we find

$$\ln A_\gamma^-(k, \omega) = \text{const} + \left( 2.03 - \frac{2}{\pi^2} \ln \frac{D_\rho}{\hbar|\omega|} \right) k^2 \quad (94)$$

at  $k \rightarrow 0$ . Thus, the spectral function should have a peak if  $\ln(D_\rho/\hbar|\omega|) > 10$ . On the other hand, the finite level spacing on the quarter-filled lattice of 228 sites<sup>22</sup> limits the frequencies such that  $\ln(D_\rho/\hbar|\omega|) \leq \ln(228/2\pi) \approx 3.6$ . Thus, to find the peak at  $k=0$ , significantly longer systems should be studied. In the spin-incoherent regime  $T \gg J$ , a similar interplay of the Gaussian peak in the spectral function with the minimum of the prefactor was discussed in Ref. 27.

## ACKNOWLEDGMENTS

The authors are grateful to K. Penc for helpful discussions and to the Aspen Center for Physics, where part of this work was done, for hospitality. This work was supported by the U.S. Department of Energy, Office of Science, under Contract No. DE-AC02-06CH11357, by Grant-in-Aid for Scientific Research (Grant No. 16GS0219) from MEXT of Japan, and by NSF DMR Grants Nos. 0237296 and 0439026.

## APPENDIX A: ANTICOMMUTATION OF OPERATORS (8)

Since electrons are fermions, in one dimension, their field operators are expected to satisfy the following anticommutation relations:

$$\psi_\gamma(x) \psi_{\gamma'}(y) + \psi_{\gamma'}(y) \psi_\gamma(x) = 0, \quad (\text{A1a})$$

$$\psi_\gamma^\dagger(x) \psi_{\gamma'}^\dagger(y) + \psi_{\gamma'}^\dagger(y) \psi_\gamma^\dagger(x) = 0, \quad (\text{A1b})$$

$$\psi_\gamma(x) \psi_{\gamma'}^\dagger(y) + \psi_{\gamma'}^\dagger(y) \psi_\gamma(x) = \delta_{\gamma\gamma'} \delta(x - y). \quad (\text{A1c})$$

Here, we check that our form of electron operators (8) is consistent with relations (A1).

We start by discussing the commutation relations of operators  $Z_{l,\gamma}$  and  $Z_{l,\gamma'}^\dagger$ . By definition, these operators act on spin chain (4) and change the number of sites as follows. Operator  $Z_{l,\gamma}$  removes site  $l$  from the spin chain if that site has spin  $\gamma$  and gives zero otherwise. Conversely, the operator  $Z_{l,\gamma}^\dagger$  adds a new site  $l$  with spin  $\gamma$  to the spin chain by inserting it between the sites  $l-1$  and  $l$ .

Let us consider the effect of operator  $Z_{l,\gamma} Z_{l',\gamma'}^\dagger$  with  $l < l'$ . This operator first removes spin  $\gamma'$  from site  $l'$  and then spin  $\gamma$  from site  $l$ . Alternatively, one can first remove site  $l$  and notice that the numbering of all sites after  $l$  has shifted by 1. Thus, to achieve the same result, at the second step one needs to remove site  $l'-1$ . We therefore conclude that

$$Z_{l,\gamma}Z_{l',\gamma'} = Z_{l'-1,\gamma'}Z_{l,\gamma} \quad l < l'. \quad (\text{A2a})$$

Repeating these arguments with operators  $Z^\dagger$ , we find

$$Z_{l,\gamma}^\dagger Z_{l',\gamma'}^\dagger = Z_{l'+1,\gamma'}^\dagger Z_{l,\gamma}^\dagger \quad l \leq l', \quad (\text{A2b})$$

$$Z_{l,\gamma}Z_{l',\gamma'}^\dagger = Z_{l'-1,\gamma'}^\dagger Z_{l,\gamma} \quad l < l'. \quad (\text{A2c})$$

Let us check Eq. (A1c) at  $x < y$ . We start by writing the first term as

$$\psi_\gamma(x)\psi_{\gamma'}^\dagger(y) = \Psi(x)Z_{l(x),\gamma}Z_{l(y),\gamma'}^\dagger\Psi^\dagger(y) \quad (\text{A3})$$

and noticing that according to definition (7), the holon created at point  $y$  is counted in  $l(y)$ , but not in  $l(x)$ . Thus,  $l(x) < l(y)$  and one can use Eq. (A2c),

$$\psi_\gamma(x)\psi_{\gamma'}^\dagger(y) = \Psi(x)Z_{l(y)-1,\gamma'}^\dagger Z_{l(x),\gamma}\Psi^\dagger(y).$$

Using definition (7) again, we find that at  $x < y$ , the operators  $Z_{l(x),\gamma}$  and  $\Psi^\dagger(y)$  commute, while  $\Psi(x)Z_{l(y)-1,\gamma'}^\dagger = Z_{l(y),\gamma'}^\dagger\Psi(x)$ . Thus, we conclude that

$$\begin{aligned} \psi_\gamma(x)\psi_{\gamma'}^\dagger(y) &= Z_{l(y),\gamma'}^\dagger\Psi(x)\Psi^\dagger(y)Z_{l(x),\gamma} \\ &= -Z_{l(y),\gamma'}^\dagger\Psi^\dagger(y)\Psi(x)Z_{l(x),\gamma} \\ &= -\psi_{\gamma'}^\dagger(y)\psi_\gamma(x), \end{aligned}$$

in agreement with Eq. (A1c). One can easily perform a similar check of Eqs. (A1a) and (A1b) and the case of  $x > y$ .

At  $x=y$ , relations (A1a) and (A1b) for operators (8) are trivially satisfied because  $\Psi(x)\Psi(x) = \Psi^\dagger(x)\Psi^\dagger(x) = 0$ . On the other hand, relation (A1c) is less straightforward. From Eq. (A3) at  $x=y$ , we get

$$\psi_\gamma(x)\psi_{\gamma'}^\dagger(x) = \Psi(x)\Psi^\dagger(x)\delta_{\gamma\gamma'}, \quad (\text{A4})$$

because the operator  $Z_{l,\gamma}Z_{l,\gamma'}^\dagger$  first creates a site with spin  $\gamma'$  and then removes the same site with spin  $\gamma$ . The second term in the left-hand side of Eq. (A1c) becomes

$$\psi_{\gamma'}^\dagger(x)\psi_\gamma(x) = \Psi^\dagger(x)\Psi(x)Z_{l(x),\gamma'}^\dagger Z_{l(x),\gamma}. \quad (\text{A5})$$

Contrary to the expectation based on Eq. (A1c),  $Z_{l,\gamma'}^\dagger Z_{l,\gamma} \neq \delta_{\gamma\gamma'}$ .

The reason for this apparent discrepancy is that our operators (8) act in a restricted Hilbert space, where two electrons cannot occupy the same point  $x$ , even if their spins are opposite. This is a fundamental feature of our theory, which reflects the fact that electrons repel each other very strongly. In this restricted space, the operator

$$\psi_\gamma(x)\psi_{\gamma'}^\dagger(x) + \psi_{\gamma'}^\dagger(x)\psi_\gamma(x) \quad (\text{A6})$$

is *not* equivalent to  $\delta_{\gamma\gamma'}$ . [Here, it is convenient to view  $x$  as a discrete coordinate and replace  $\delta(x-x) \rightarrow 1$  in Eq. (A1c)]. Indeed, in our Hilbert space, the state at point  $x$  can be either empty or occupied with a single electron, with possible spins  $\uparrow$  or  $\downarrow$ . When acting on these states, operator (A6) has the following effect:

$$|0\rangle \rightarrow \delta_{\gamma\gamma'},$$

$$|\uparrow\rangle \rightarrow \delta_{\gamma,\uparrow}|\gamma'\rangle,$$

$$|\downarrow\rangle \rightarrow \delta_{\gamma,\downarrow}|\gamma'\rangle.$$

It is easy to check that the sum of operators (A4) and (A5) has exactly the same effect on these three states. Thus, our expressions (8) for the electron operators have correct anti-commutation relations.

## APPENDIX B: BOSONIZATION OF THE OPERATORS $Z_{L,\gamma}$ IN THE CASE OF NONVANISHING MAGNETIZATION

In Sec. III B, we bosonized the operators  $Z_{l,\gamma}$  in the SU(2)-symmetric case, when no magnetic field is applied to the system. In the presence of the field, the ground state of the system has unequal densities of electrons with spins  $\uparrow$  and  $\downarrow$ , giving rise to a finite magnetization  $m=2\langle S_z^+ \rangle$ . As a result, the Fermi sea of the Jordan-Wigner fermions (Fig. 2) expands to accommodate their increased density  $\langle a_l^\dagger a_l \rangle = (1+m)/2$  [see Eq. (19)]. The Fermi points corresponding to this density are

$$q_L = \frac{\pi}{2}(1-m), \quad q_R = \frac{\pi}{2}(3+m). \quad (\text{B1})$$

Our bosonization procedure in Sec. III B was performed for  $m=0$ , but the derivation can be easily generalized to the case of  $m>0$  by using the proper values [Eq. (B1)] of  $q_{L,R}$  instead of Eq. (22). The resulting bosonization expressions for the operators  $Z_{l,\gamma}$  are given by

$$\begin{aligned} Z_{l,\gamma} &= e^{i(\pi/2)(1\mp m)l} e^{\mp i[\varphi(l)+(1/2)(1\mp m)\vartheta(l)]} \\ &\quad + e^{-i(\pi/2)(1\mp m)l} e^{\pm i[\varphi(l)-(1/2)(1\mp m)\vartheta(l)]}, \end{aligned} \quad (\text{B2})$$

with the upper and lower signs corresponding to  $\gamma=\uparrow$  and  $\downarrow$ , respectively.

It is instructive to substitute Eq. (B2) into our expression (14) for the electron annihilation operators and compare the resulting bosonization formulas with the standard expressions (15). As we discussed in Sec. III C, the right-moving electron is constructed out of a right-moving holon and a left-moving spinon. We therefore combine the second term in Eq. (B2) with the first term in the parentheses in Eq. (14) and obtain

$$\psi_{R\gamma}(x) = \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} e^{i[(1\pm m)/2][k_F^h x + \phi(x)]} e^{\pm i[\varphi(nx) - [(1\mp m)/2]\vartheta(nx)]}. \quad (\text{B3})$$

This result should be compared with the standard bosonization expression (15a).

As expected, instead of  $k_F = k_F^h/2$ , we find that the Fermi momentum is now a function of the magnetization:

$$k_{F\uparrow} = \frac{1+m}{2}k_F^h, \quad k_{F\downarrow} = \frac{1-m}{2}k_F^h. \quad (\text{B4})$$

In addition, by comparing Eq. (B3) and the analogous expression for the left-moving electron,

$$\psi_{L\gamma}(x) = \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} e^{-i[(1\pm m)/2][k_F^h x + \phi(x)]} e^{\mp i\{\varphi(nx) + [(1\mp m)/2]\vartheta(nx)\}}, \quad (\text{B5})$$

with Eq. (15) one finds the following relations between the bosonic fields:

$$\phi(x) = \sqrt{2}\phi_\rho(x), \quad (\text{B6})$$

$$\theta(x) = \frac{\theta_\rho(x) + m\theta_\sigma(x)}{\sqrt{2}}, \quad (\text{B7})$$

$$\varphi(l) = \frac{\phi_\sigma(l/n) - m\phi_\rho(l/n)}{\sqrt{2}}, \quad (\text{B8})$$

$$\vartheta(l) = \sqrt{2}\theta_\sigma(l/n). \quad (\text{B9})$$

These relations generalize our earlier expressions (18a), (18b), (29a), and (29b) to the case of nonvanishing magnetization. It is worth noting that at  $m > 0$ , the original charge and spin boson modes are mixed. This mixing was discussed in Ref. 52.

#### APPENDIX C: SOLUTION OF EQUATION (57) WITH CONDITIONS (61a) and (61b)

In this appendix, we show that the solution of Eq. (57) with conditions (61a) and (61b) has the form of Eqs. (62a) and (62b). We first rewrite integral equation (57) in terms of complex variables  $w = e^{iq}$  and  $z = e^{iq'}$ . Using the fact that  $c_\gamma^+$  is an even function of  $q$ , we find

$$c^-(w) = \frac{w^2 + 1}{2w} c^+(w) - \frac{w}{2\pi i} \oint \frac{dz}{z-w} \frac{z^2 - 1}{z^2} c^+(z). \quad (\text{C1})$$

Here, we use the notations  $c_\gamma^\pm(q) = c^\pm(w)$ ; the integral is taken over the unit circle  $|z|=1$  in counterclockwise direction.

According to condition (61b), function  $c^-(w)$  vanishes when  $w$  is on the right semicircle  $R$  (defined as  $w = e^{iq}$  with  $q$  between  $-\pi/2$  and  $\pi/2$ ). In addition,  $c^+(z)$  vanishes on the left semicircle [see Eq. (61a)]. Thus, for  $w \in R$ , we have

$$\frac{w^2 + 1}{w^2} c^+(w) = \frac{1}{\pi i} \int_R \frac{dz}{z-w} \frac{z^2 - 1}{z^2} c^+(z), \quad w \in R. \quad (\text{C2})$$

Here, we assume that the contour  $R$  is traversed in the counterclockwise direction, from  $z = -i$  to  $z = i$ .

This equation can be solved using the theory of singular integral equations.<sup>43</sup> It will be convenient to introduce a new unknown function

$$\phi(z) = \frac{z^2 - 1}{z^2} c^+(z). \quad (\text{C3})$$

Then, integral equation (C2) takes the form

$$\frac{w^2 + 1}{w^2 - 1} \phi(w) = \frac{1}{\pi i} \int_R dz \frac{\phi(z)}{z-w}, \quad w \in R. \quad (\text{C4})$$

In solving this equation, we will assume that the unknown function  $\phi(w)$  is analytic with possible exception of inte-

grable singularities at the ends  $z = \pm i$  of the contour  $R$ . Let us introduce a new function

$$\Phi(w) = \frac{1}{2\pi i} \int_R dz \frac{\phi(z)}{z-w}. \quad (\text{C5})$$

Obviously,  $\Phi(w)$  is analytic everywhere except the contour  $R$  and approaches zero at infinity. At the contour,  $\Phi(w)$  has a branch cut. When  $w$  approaches  $R$  from the left or right, the function  $\Phi(w)$  takes the values

$$\Phi^\pm(w) = \frac{1}{2\pi i} \int_R dz \frac{\phi(z)}{z-w \pm \delta}, \quad \delta \rightarrow +0. \quad (\text{C6})$$

One can easily see that for  $w \in R$ , we have

$$\Phi^+(w) - \Phi^-(w) = \phi(w), \quad (\text{C7})$$

$$\Phi^+(w) + \Phi^-(w) = \frac{1}{\pi i} \int_R dz \frac{\phi(z)}{z-w}. \quad (\text{C8})$$

Substituting these relations into Eq. (C4), we find that the values of  $\phi(w)$  on the two sides of the branch cut satisfy the following linear relation:

$$\Phi^+(w) = w^2 \Phi^-(w), \quad w \in R. \quad (\text{C9})$$

An analytic function that satisfies these conditions on the two sides of the contour  $R$ , falls off to zero at infinity, and does not diverge faster than  $1/(w \pm i)$  at the ends of the contour is unique up to an arbitrary numerical coefficient. It can be found using the techniques discussed in Sec. 79 of Ref. 43. The solution is

$$\Phi(w) = \frac{1}{w^2 + 1} \exp\left(\frac{1}{\pi i} \int_R \frac{\ln z}{z-w} dz\right), \quad (\text{C10})$$

where the logarithm is defined with the branch cut along the negative real axis. It is easy to check directly that function (C10) does satisfy the above conditions.

For  $w \in R$ , by combining Eqs. (C3), (C7), and (C9), one finds  $c^+(w) = \Phi^+(w)$ . Then, substituting  $w = e^{iq}$ , one finds result (62a).

Our next goal is to find  $c^-(w)$  for  $w$  on the left unit semicircle  $L$ , defined by  $w = e^{iq}$  with  $q$  between  $\pi/2$  and  $3\pi/2$ . Using Eq. (C1) and noticing that  $c^+(w) = 0$  on  $L$ , we find

$$c^-(w) = -\frac{w}{2\pi i} \int_R \frac{dz}{z-w} \frac{z^2 - 1}{z^2} c^+(z) = -w\Phi(w) \quad (\text{C11})$$

[see Eqs. (C3) and (C5)]. Then, substituting  $w = e^{iq}$ , we obtain result (62b) with the same normalization constant as in Eq. (62a).

#### APPENDIX D: MAPPING OF THE BOSONIZED HEISENBERG SPIN CHAIN TO THE SPIN SECTOR OF WEAKLY INTERACTING ELECTRON SYSTEM

##### 1. Consequences of the spin-rotation symmetry

In Sec. IV C 3, we have evaluated the first correction (73) to the correlator  $c_\gamma^\pm(q)$  above the Fermi point  $q = \pi/2$ , where

the simple bosonization result (67a) vanishes. The correction originated from two sources. First, we found the contribution due to the deviation of the quadratic part of the Hamiltonian from the fixed point,  $\mathcal{K} \neq 1/2$ , and then we included sine-Gordon term (28b). The latter correction turned out to be larger than the former one by a factor of 2. Here, we show that this is a result of the spin-rotation symmetry of the problem.

To this end, we utilize the equivalence of the bosonized Hamiltonians of Heisenberg spin chain (28) and spin part (16b) of the Hamiltonian of weakly interacting electrons. The exact form of the electron-electron interactions does not affect the general form of Hamiltonian (16). In the simplest case, one can consider only backscattering of electrons by each other. For the electrons in the vicinity of the Fermi level, the most general form of backscattering Hamiltonian is

$$g_{\gamma\gamma'\delta\delta'} \psi_{L\gamma}^\dagger \psi_{R\delta}^\dagger \psi_{L\gamma'} \psi_{R\delta'}, \quad (\text{D1})$$

with the coupling constant

$$g_{\gamma\gamma'\delta\delta'} = \frac{1}{2} g_1 (\delta_{\gamma\gamma'} \delta_{\delta\delta'} + \boldsymbol{\sigma}_{\gamma\gamma'} \cdot \boldsymbol{\sigma}_{\delta\delta'}), \quad (\text{D2})$$

where  $\boldsymbol{\sigma}_{\gamma\gamma'}$  is the vector of standard Pauli matrices and  $g_1$  is the  $2k_F$  Fourier component of the interaction potential. In Eq. (D1), the summation over repeating spin indices is implied.

The exact form (D1), (D2) of the backscattering term in the Hamiltonian is dictated by the SU(2) symmetry of the problem with respect to the rotation of electron spins. Ignoring this symmetry for the moment, we will view Eq. (D2) as a special case of tensor

$$g_{\gamma\gamma'\delta\delta'} = \frac{1}{2} (g_{1\rho} \delta_{\gamma\gamma'} \delta_{\delta\delta'} + g_{1z} \sigma_{\gamma\gamma'}^z \sigma_{\delta\delta'}^z + g_{1x} \sigma_{\gamma\gamma'}^x \sigma_{\delta\delta'}^x + g_{1y} \sigma_{\gamma\gamma'}^y \sigma_{\delta\delta'}^y). \quad (\text{D3})$$

Unlike Eq. (D2), this form of coupling violates the spin-rotation symmetry, unless  $g_{1x} = g_{1y} = g_{1z}$ . The standard treatments of weakly interacting electron systems, including the derivation of bosonized Hamiltonian (16), start with two constants,  $g_{1\parallel} = g_{1z}$  and  $g_{1\perp} = g_{1x} = g_{1y}$ , and eventually equate  $g_{1\parallel}$  and  $g_{1\perp}$ .

Interaction constants  $g_{1\rho}$ ,  $g_{1\parallel}$ , and  $g_{1\perp}$  affect different terms of bosonized Hamiltonian (16). Parameter  $g_{1\rho}$  corresponds to density-density coupling and affects the Hamiltonian of the charge degrees of freedom via renormalization of  $v_\rho$  and  $K_\rho$ . It does not affect  $H_\sigma$  and for our purposes can be ignored. The coupling constant  $g_{1\parallel}$  enters via  $K_\sigma = 1 + g_{1\parallel}/2$ , whereas the spin flip scattering accounted for by  $g_{1\perp}$  transforms to the sine-Gordon term in Eq. (16b).

Our calculation of correction (73) neglected the sine-Gordon term. Thus,  $g^2$  in Eq. (73) is, in fact,  $g_{1z}^2$ . Since  $c_\gamma^+(q)$  is invariant with respect to spin rotations, identical contributions should be expected from coupling constants  $g_{1x}$  and  $g_{1y}$ . While evaluating the correction to  $c_\gamma^+(q)$  due to sine-Gordon term (28b), we accounted for both  $g_{1x}$  and  $g_{1y}$ , and, as expected, obtained twice the result (73).

## 2. Alternative evaluation of the correlators $c_\gamma^+(q)$ near the Fermi points

The mapping of bosonized Hamiltonian (28) of the Heisenberg spin chain and the Hamiltonian  $H_\sigma$ , Eq. (16b), describing the dynamics of the spin sector of weakly interacting electron gas, enables one to obtain an alternative expression for the correlators  $c_\gamma^+(q)$  near the Fermi point. Let us consider the Green's function of right-moving electrons  $G_R(x, t)$  traced over the spin indices. It is well known<sup>20</sup> that asymptotically at large  $x$  and  $t$ , it separates into a product of charge and spin factors,

$$G_R(x, t) = \frac{1}{\pi} g_\rho(x, t) g_\sigma(x, t). \quad (\text{D4})$$

The two factors are most easily computed using bosonization transformation (15), in which case the charge and spin factors are obtained by averaging the exponentials of bosonic fields  $\phi_\rho$ ,  $\theta_\rho$  and  $\phi_\sigma$ ,  $\theta_\sigma$  in Eq. (15a), respectively. In the absence of electron-electron interactions, the parameters of Hamiltonian (16) take unperturbed values  $v_\rho = v_\sigma = v_F$ ,  $K_\rho = K_\sigma = 1$ , and  $g_{1\perp} = 0$ . Then, one finds

$$g_\rho^{(0)}(x, t) = g_\sigma^{(0)}(x, t) = \frac{1}{(x - v_F t + i \delta \operatorname{sgn} t)^{1/2}}, \quad (\text{D5})$$

and Eq. (D4) reproduces the standard expression for the Green's function of noninteracting electrons.

In the presence of interactions, the parameters of Hamiltonian (16) renormalize, and the charge and spin components of the Green's function show nontrivial behavior. We showed in Sec. III C that the exponentials of the bosonic fields  $\phi_\sigma$  and  $\theta_\sigma$  in Eq. (15a) are equivalent to bosonized expression (41) for the operators  $Z_{l,\gamma}$ . Thus, the correlator  $\langle Z_{l,\gamma} Z_{0,\gamma}^\dagger \rangle$  in definition (51a) of  $c_\gamma^+(q)$  can be found from  $g_\sigma(x, t)$  at  $x = l/n$  and  $t = 0$ .

We will find  $g_\sigma(x, t)$  by calculating electron Green's function (D4) and identifying its spin component. Since Hamiltonian (16b) of the spin sector is universal at low energies, the specific form of the electron-electron interactions is not important. It is most convenient to choose form (D1) with coupling constant (D3) chosen so that  $g_{1\rho} = 0$  and  $g_{1x} = g_{1y} = g_{1z} = g_1$ . In this case, the interactions do not affect the charge sector, and  $g_\rho(x, t)$  retains its unperturbed value (D5), and

$$g_\sigma(x, t) = \pi (x - v_F t + i \delta \operatorname{sgn} t)^{1/2} G_R(x, t). \quad (\text{D6})$$

The electronic Green's function can be studied using straightforward perturbation theory in the coupling constant  $g_1$ .

The first nonvanishing contribution to  $G_R(x, t)$  appears in the second order and is given by the diagram in Fig. 4. The respective correction to the Green's function is

$$\delta G_R(k, \omega) = 2 G_R^{(0)}(k, \omega) \Sigma(k, \omega) G_R^{(0)}(k, \omega), \quad (\text{D7})$$

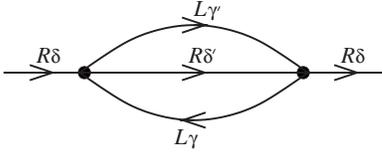


FIG. 4. The second-order diagram for the perturbative calculation of the electronic Green's function  $G_R(x, t)$ .

where the factor of 2 accounts for the trace over spin variables and the self-energy

$$\Sigma(k, \omega) = \frac{3g_1^2}{2} \int \frac{dk_1 dk_2 d\omega_1 d\omega_2}{(2\pi)^4} G_L^{(0)}(k_1, \omega_1) G_L^{(0)}(k_2, \omega_2) \times G_R^{(0)}(k + k_1 - k_2, \omega + \omega_1 - \omega_2). \quad (\text{D8})$$

Here, the factor  $3g_1^2$  appears as the sum  $g_{1x}^2 + g_{1y}^2 + g_{1z}^2$ . Substituting the unperturbed Green's functions

$$G_L^{(0)}(k, \omega) = \frac{1}{\omega + v_F k - i\delta \operatorname{sgn} k}, \quad (\text{D9a})$$

$$G_R^{(0)}(k, \omega) = \frac{1}{\omega - v_F k + i\delta \operatorname{sgn} k}, \quad (\text{D9b})$$

and performing integration with respect to  $\omega_1$ ,  $\omega_2$ , and one of the momenta, we find

$$\Sigma(k, \omega) = \Sigma_v(k) + \tilde{\Sigma}(k, \omega), \quad (\text{D10})$$

where

$$\Sigma_v(k) = -\frac{3g_1^2 k}{16\pi^2 v_F}, \quad (\text{D11})$$

$$\tilde{\Sigma}(k, \omega) = \frac{3g_1^2(\omega - v_F k)}{16\pi^2 v_F} \int_0^\infty \left( \frac{\theta(k+q)}{\omega - v_F k - 2v_F q + i\delta} - \frac{\theta(-k+q)}{\omega - v_F k + 2v_F q - i\delta} \right) e^{-\alpha q} dq. \quad (\text{D12})$$

Here, we have introduced the short-distance cutoff  $\alpha$  for the electron-electron interactions. In principle, this cutoff may not coincide with the bandwidth cutoff  $\alpha$  used in the bosonization procedure. This distinction is not important for the present discussion.

The two second-order contributions to the electron self-energy have very different meanings. The term  $\Sigma_v$  accounts for a small correction to the velocity of spin excitations, which for our purposes can be ignored. On the other hand,  $\tilde{\Sigma}$  leads to the logarithmic renormalization of the electron Green's function, which affects the singular behavior of the correlators  $c_\gamma^\pm(q)$  near the Fermi points. We therefore explore this correction in more detail.

#### a. Logarithmic correction to the Green's function

Let us now substitute expression (D12) for the self-energy  $\tilde{\Sigma}(k, \omega)$  in Eq. (D7) and perform the Fourier transformation

to  $x$  and  $t$  variables. The resulting correction to the Green's function has the form

$$\delta G_R(x, t) = \frac{3g_1^2}{32\pi^3 v_F^2} \frac{1}{x - v_F t + i\delta \operatorname{sgn} t} \ln \frac{\alpha^2}{(\alpha + iv_F |t|)^2 + x^2}. \quad (\text{D13})$$

This expression is consistent with the logarithmic renormalization of the electron Green's function studied earlier in the  $g$ -ology theory [cf. Eq. (4.24) of Ref. 45]. [Our Eq. (D13) is obtained by neglecting the constant  $g_2$  and replacing  $4g_1^2$  with  $3g_1^2$  to account for the fact that we assume  $g_{1\rho}=0$ .]

We can now separate the spin component  $g_\sigma(x, t)$  of the electron Green's function  $G_R(x, t)$  following prescription (D6) and obtain

$$g_\sigma(x, t) = \frac{1}{(x - v_F t + i\delta \operatorname{sgn} t)^{1/2}} \times \left( 1 - \frac{3y_1^2}{32} \ln \frac{(\alpha + iv_F |t|)^2 + x^2}{\alpha^2} \right), \quad (\text{D14})$$

where  $y_1 = g_1 / \pi v_F$ . As expected,  $g_s(x, +0)$  reproduces the logarithmic correction to the correlator  $\langle Z_{l\gamma} Z_{0,\gamma}^\dagger \rangle$  used in Eq. (72), with the additional factor of 3 correctly included.

#### b. Renormalization of the Green's function

Logarithmic correction (D14) to the Green's function grows at long distances and can, in principle, become large despite the smallness of the prefactor  $y_1^2$ . To find out whether this is the case, one can compute the Green's function in the leading logarithm approximation. We accomplish this by adopting the multiplicative renormalization procedure of Ref. 45. We present the spin component of the Green's function as

$$g_\sigma(x, t) = d(\xi) g_\sigma^{(0)}(x, t). \quad (\text{D15})$$

Here, the new function  $d(\xi)$  is expected to depend on  $x$  and  $t$  very slowly, via their logarithm,

$$\xi(x, t) = \frac{1}{2} \ln \frac{(\alpha + iv_F |t|)^2 + x^2}{\alpha^2}. \quad (\text{D16})$$

In particular, our result (D14) has form (D15) with

$$d(\xi) = 1 - \frac{3y_1^2}{16} \xi. \quad (\text{D17})$$

As we increase  $\xi$ , correction (D17) grows and may no longer remain small. In addition, the coupling constant  $y_1$  itself depends on  $\xi$  as

$$y_1(\xi) = \frac{y_1}{1 + y_1 \xi} \quad (\text{D18})$$

(see Ref. 45).

Following the general prescription<sup>45</sup> to account for the multiplicative corrections to the Green's function, we use Eq. (D17) to write the renormalization group equation upon  $\ln d$ ,

$$\frac{d}{d\xi} \ln d(\xi) = -\frac{3}{16} y_1^2(\xi). \quad (\text{D19})$$

Solution of this equation with  $y_1(\xi)$  given by Eq. (D18) has the form

$$d(\xi) = \exp\left(-\frac{3}{16} \frac{y_1^2 \xi}{1 + y_1 \xi}\right). \quad (\text{D20})$$

At small  $y_1 \xi$ , it reproduces perturbative expansion (D17), while in the limit  $\xi \rightarrow \infty$ , we obtain a finite renormalization of the Green's function,

$$g_\sigma(x, t) = \exp\left(-\frac{3y_1}{16}\right) \frac{1}{(x - v_F t + i\delta \operatorname{sgn} t)^{1/2}}. \quad (\text{D21})$$

In a weakly interacting electron gas, the coupling constant  $y_1 \ll 1$ , and renormalization (D21) can be ignored. As interactions become stronger,  $y_1$  increases and reaches values of order unity. Thus, one can expect that the coefficient  $\chi$  in asymptotes (75a) and (75b) of the correlators  $c_\gamma^\pm(q)$  near the Fermi point will slowly decrease from its numerically obtained value  $\chi \sim 0.8$  to a significantly lower number at  $q \rightarrow \pi/2$ .

- <sup>1</sup>S. Tarucha, T. Honda, and T. Saku, *Solid State Commun.* **94**, 413 (1995).
- <sup>2</sup>A. Yacoby, H. L. Stormer, N. S. Wingreen, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **77**, 4612 (1996).
- <sup>3</sup>K. J. Thomas, J. T. Nicholls, M. Y. Simmons, M. Pepper, D. R. Mace, and D. A. Ritchie, *Phys. Rev. Lett.* **77**, 135 (1996).
- <sup>4</sup>K. J. Thomas, J. T. Nicholls, N. J. Appleyard, M. Y. Simmons, M. Pepper, D. R. Mace, W. R. Tribe, and D. A. Ritchie, *Phys. Rev. B* **58**, 4846 (1998).
- <sup>5</sup>S. M. Cronenwett, H. J. Lynch, D. Goldhaber-Gordon, L. P. Kouwenhoven, C. M. Marcus, K. Hirose, N. S. Wingreen, and V. Umansky, *Phys. Rev. Lett.* **88**, 226805 (2002).
- <sup>6</sup>A. Kristensen, H. Bruus, A. E. Hansen, J. B. Jensen, P. E. Lindelof, C. J. Marckmann, J. Nygard, C. B. Sørensen, F. Beuscher, A. Forchel, and M. Michel, *Phys. Rev. B* **62**, 10950 (2000).
- <sup>7</sup>L. P. Rokhinson, L. N. Pfeiffer, and K. W. West, *Phys. Rev. Lett.* **96**, 156602 (2006).
- <sup>8</sup>R. Crook, J. Prance, K. J. Thomas, S. J. Chorley, I. Farrer, D. A. Ritchie, M. Pepper, and C. G. Smith, *Science* **312**, 1359 (2006).
- <sup>9</sup>K. J. Thomas, J. T. Nicholls, M. Pepper, W. R. Tribe, M. Y. Simmons, and D. A. Ritchie, *Phys. Rev. B* **61**, R13365 (2000).
- <sup>10</sup>D. J. Reilly, G. R. Facer, A. S. Dzurak, B. E. Kane, R. G. Clark, P. J. Stiles, J. L. O'Brien, N. E. Lumpkin, L. N. Pfeiffer, and K. W. West, *Phys. Rev. B* **63**, 121311(R) (2001).
- <sup>11</sup>O. M. Auslaender, A. Yacoby, R. de Picciotto, K. W. Baldwin, L. N. Pfeiffer, and K. W. West, *Phys. Rev. Lett.* **84**, 1764 (2000).
- <sup>12</sup>M. Bockrath, D. H. Cobden, J. Lu, A. G. Rinzler, R. E. Smalley, L. Balents, and P. L. McEuen, *Nature (London)* **397**, 598 (1999).
- <sup>13</sup>Z. Yao, H. W. C. Postma, L. Balents, and C. Dekker, *Nature (London)* **402**, 273 (1999).
- <sup>14</sup>F. D. M. Haldane, *J. Phys. C* **14**, 2585 (1981).
- <sup>15</sup>T. Giamarchi, *Quantum Physics in One Dimension* (Clarendon, Oxford, 2004).
- <sup>16</sup>Other features of the Luttinger-liquid behavior have been observed in experiments on spin chains and organic conductors (see Ref. 15 and references therein).
- <sup>17</sup>D. L. Maslov and M. Stone, *Phys. Rev. B* **52**, R5539 (1995).
- <sup>18</sup>V. V. Ponomarenko, *Phys. Rev. B* **52**, R8666 (1995).
- <sup>19</sup>I. Safi and H. J. Schulz, *Phys. Rev. B* **52**, R17040 (1995).
- <sup>20</sup>I. E. Dzyaloshinskii and A. I. Larkin, *Sov. Phys. JETP* **38**, 202 (1974).
- <sup>21</sup>K. Penc, F. Mila, and H. Shiba, *Phys. Rev. Lett.* **75**, 894 (1995).
- <sup>22</sup>K. Penc, K. Hallberg, F. Mila, and H. Shiba, *Phys. Rev. Lett.* **77**, 1390 (1996).
- <sup>23</sup>K. Penc, K. Hallberg, F. Mila, and H. Shiba, *Phys. Rev. B* **55**, 15475 (1997).
- <sup>24</sup>V. V. Cheianov and M. B. Zvonarev, *Phys. Rev. Lett.* **92**, 176401 (2004).
- <sup>25</sup>V. V. Cheianov and M. B. Zvonarev, *J. Phys. A* **37**, 2261 (2004).
- <sup>26</sup>G. A. Fiete and L. Balents, *Phys. Rev. Lett.* **93**, 226401 (2004).
- <sup>27</sup>G. A. Fiete, J. Qian, Y. Tserkovnyak, and B. I. Halperin, *Phys. Rev. B* **72**, 045315 (2005).
- <sup>28</sup>K. A. Matveev, *Phys. Rev. Lett.* **92**, 106801 (2004).
- <sup>29</sup>K. A. Matveev, *Phys. Rev. B* **70**, 245319 (2004).
- <sup>30</sup>G. A. Fiete, *Rev. Mod. Phys.* **79**, 801 (2007).
- <sup>31</sup>For a review of the recent results for electronic transport in the spin-incoherent regime, see Ref. 30.
- <sup>32</sup>K. A. Matveev, A. Furusaki, and L. I. Glazman, *Phys. Rev. Lett.* **98**, 096403 (2007).
- <sup>33</sup>M. Ogata and H. Shiba, *Phys. Rev. B* **41**, 2326 (1990).
- <sup>34</sup>E. Lieb and D. Mattis, *Phys. Rev.* **125**, 164 (1962).
- <sup>35</sup>W. Häusler, *Z. Phys. B: Condens. Matter* **99**, 551 (1996).
- <sup>36</sup>A. D. Klironomos, R. R. Ramazashvili, and K. A. Matveev, *Phys. Rev. B* **72**, 195343 (2005).
- <sup>37</sup>M. M. Fogler and E. Pivovarov, *Phys. Rev. B* **72**, 195344 (2005).
- <sup>38</sup>Similar physics results in the conclusion that the spin excitations of a one-dimensional electron system may carry electric charge (Refs. 53 and 54).
- <sup>39</sup>F. D. M. Haldane, *Phys. Rev. Lett.* **47**, 1840 (1981).
- <sup>40</sup>K. Penc and M. Serhan, *Phys. Rev. B* **56**, 6555 (1997).
- <sup>41</sup>S. Sorella and A. Parola, *J. Phys.: Condens. Matter* **4**, 3589 (1992).
- <sup>42</sup>The same correlation function in coordinate representation was first introduced and studied numerically by exact diagonalization of spin chains of 26 sites by Ogata and Shiba (Ref. 33). (Our notation  $\langle Z_{l\sigma}^\dagger Z_{0\sigma} \rangle$  corresponds to  $2\omega_{0 \rightarrow l}$  in Ref. 33.)
- <sup>43</sup>N. I. Muskhvelishvili, *Singular Integral Equations* (Dover, New York, 1992).
- <sup>44</sup>K. Penc (private communication).
- <sup>45</sup>J. Sólyom, *Adv. Phys.* **28**, 201 (1979).
- <sup>46</sup>V. Meden and K. Schönhammer, *Phys. Rev. B* **46**, 15753 (1992).
- <sup>47</sup>J. Voit, *Phys. Rev. B* **47**, 6740 (1993).

- <sup>48</sup>A. P. Kampf and J. R. Schrieffer, *Phys. Rev. B* **42**, 7967 (1990).
- <sup>49</sup>O. M. Auslaender, A. Yacoby, R. de Picciotto, K. W. Baldwin, L. N. Pfeiffer, and K. W. West, *Science* **295**, 825 (2002).
- <sup>50</sup>O. M. Auslaender, H. Steinberg, A. Yacoby, Y. Tserkovnyak, B. I. Halperin, K. W. Baldwin, L. N. Pfeiffer, and K. W. West, *Science* **308**, 88 (2005).
- <sup>51</sup>G. D. Mahan, *Many-Particle Physics*, 2nd ed. (Plenum, New York, 1990), Sec. 8.3.
- <sup>52</sup>T. Hikiyara, A. Furusaki, and K. A. Matveev, *Phys. Rev. B* **72**, 035301 (2005).
- <sup>53</sup>S. Brazovskii, S. Matveenko, and P. Nozières, *J. Phys. I* **4**, 571 (1994).
- <sup>54</sup>C. Nayak, K. Shtengel, D. Orgad, M. P. A. Fisher, and S. M. Girvin, *Phys. Rev. B* **64**, 235113 (2001).